A CRITERION FOR ALMOST ALTERNATING LINKS TO BE NON-SPLITTABLE

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1. Introduction

The notion of almost alternating links was introduced by C. Adams et al ([1]). Here we give a sufficient condition for an almost alternating link diagram to represent a non-splittable link. This solves a question asked in [1]. A partial solution for special almost alternating links has been obtained by M. Hirasawa ([4]). As its applications, Theorem 2.3 gives us a way to see if a given almost alternating link diagram represents a splittable link without increasing numbers of crossings of diagrams in the process. Moreover, we show that almost alternating links with more than two components are non-trivial. In Section 2, we state them in detail. To show our theorem, we basically use a technique invented by W. Menasco (see [5, 6]). We review it in Section 3. However, we also apply " charge and discharge method" to our graph-theoretic argument, which is used to prove the four color theorem in [3].

2. The main theorem and its applications

Menasco has shown that an alternating link diagram can represent a splittable link only in a trivial way.

Theorem 2.1. ([5]) *If a link L has a connected alternating diagram, then L is non-splittable.*

We say a link diagram $\widetilde{L}$ on S^2 is *almost alternating* if one crossing change makes $\widetilde{L}$ alternating. A link L is *almost alternating* if L is not alternating and L has an almost alternating diagram. We call a crossing of an almost alternating diagram a *dealternator* if the crossing change at the crossing makes the diagram alternating. An almost alternating diagram may have more than one dealternator. However, we can uniquely decide a connected almost alternating diagram if the diagram has more than one dealternater (Proposition 2.2). Since the statement of Proposition 2.2 does not contradict the statement of Theorem 2.3, we may assume that our almost alternating diagram has exactly one dealternator from now on.

Proposition 2.2. *If a connected almost alternating diagram $\widetilde{L}$ has more than one dealternator, then $\widetilde{L}$ is a diagram obtained from a Hopf link diagram with two crossings by changing one of the crossings.*

Proof. Assume that $\widetilde{L}$ has more than one dealternator. Let α be one of the dealternators. Then, α is adjacent to other four crossings (some of them may be the same). Let β be another dealternator. Since the crossing change at β makes $\widetilde{L}$ alternating, each of those four crossings must coinsides β. Then, $\widetilde{L}$ is a diagram obtained from a Hopf link diagram with two crossings by changing one of the crossings. □

A diagram $\widetilde{L}$ on S^2 is *prime* if $\widetilde{L}$ is connected, $\widetilde{L}$ has at least one crossing, and there does not exist a simple closed curve on S^2 meeting $\widetilde{L}$ transversely in just two points belonging to different arcs of $\widetilde{L}$. If

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our almost alternating diagram is non-prime, then it is a connected sum of a prime almost alternating diagram and alternating diagrams. Therefore, we may restrict our interest to prime diagrams, since we know that a connected alternating diagram represents a non-splittable link (Theorem 2.1).

In this paper, we say a diagram is *reduced* if it is not diagram I or diagram II in Figure 1, where we allow both two types of crossings which give almost alternating diagrams with the dealternator at the marked double point. Then our main theorem is the following.

Theorem 2.3. *If a link L has a connected, prime, reduced almost alternating diagram, then L is non-splittable.*

FIGURE 1. Non reduced diagrams

Theorem 2.3 gives us a way to see if a given almost alternating diagram $\widetilde{L}$ represents a splittable link or not; here we may assume that $\widetilde{L}$ is connected and prime. If $\widetilde{L}$ is reduced, then it represents a non-splittable link from Theorem 2.3. If $\widetilde{L}$ is not reduced, then it is diagram I or diagram II in Figure 1. If it is diagram I, then we obtain the diagram with no crossings of the trivial link with two components or an alternating diagram from $\widetilde{L}$ by applying reducing move I to $\widetilde{L}$ (see Figure 2). Then, we can see if $\widetilde{L}$ represents a splittable link or not from Theorem 2.1. If it is diagram II, then we obtain another connected, prime almost alternating diagram $\widetilde{L}'$ from $\widetilde{L}$ with less crossings than those of $\widetilde{L}$ by applying reducing move II to $\widetilde{L}$. Then, we can see if $\widetilde{L}$ represents a splittable link or not by continuing this process as far as we have diagram II.

FIGURE 2. Reducing moves

As a corollary of Theorem 2.3, we immediately have the following.

Corollary 2.4. *If a link L has a connected, prime, reduced almost alternating diagram, then L is non-trivial.*

Moreover, we obtain the following if we restrict our interest to links with more than two components.

Corollary 2.5. *If a link L with more than two components has a connected almost alternating diagram, then L is non-trivial.*

Proof. Suppose that there exists a connected almost alternating diagram $\widetilde{L}$ of the trivial link L with n (> 2) components. Here we assume that $\widetilde{L}$ has the minimal crossings among such diagrams, and then $\widetilde{L}$ is prime. Since L is also splittable, $\widetilde{L}$ is not reduced from Theorem 2.3. If $\widetilde{L}$ is diagram II, we obtain another almost alternating link diagram of L with less crossings than those of $\widetilde{L}$ by applying reducing move II. This contradicts the minimality of the number of crossings of $\widetilde{L}$. Therefore, $\widetilde{L}$ is diagram I. Then apply reducing move I to $\widetilde{L}$. If we obtain a connected diagram, then it is a connected alternating link diagram, which is non-splittable. Thus, we have a disconnected alternating diagram consisting of two connected components. Since L has more than two components, at least one of them has more than one component, which is a connected alternating link diagram and thus non-splittable. Thus, L is non-trivial, which contradicts our assumption. $\square$

3. Standard Position for A Splitting Sphere

Let $\widetilde{L} \subset \mathbb{R}^2 \subset S^2 = \mathbb{R}^2 \cup \{\infty\}$ be a diagram of a splittable link L. Here, we do not assume that $\widetilde{L}$ is almost alternating. Note that we may speak sensibly about points "above" or "below" $\widetilde{L}$ and also about "inside" or "outside" of some reasion, since we consider the projection plane $\mathbb{R}^2$ as a subspace of a 2-sphere in S^3.

Following [5] and [6], put a 3-ball, called a *crossing ball*, at each crossing point of $\widetilde{L}$. Then, isotope L so that, at each crossing point, the overstrand runs on the upper hemisphere and the understrand runs on the lower hemisphere as shown in Figure 3. We call the boundary of such a 3-ball a *bubble*.

Figure 3. A crossing ball

Let S_+ (resp. S_-) be the 2-sphere obtained from S^2 by replacing the intersection disk of each crossing ball and S^2 by the upper (resp. lower) hemisphere of the bubble of the crossing ball. We will use the notation $S_\pm$ to mean S_+ or S_- and similarly for other symbols with subscript $\pm$.

Let $F \in S^3 - L$ be a splitting sphere for a splittable link L. We may isotope F to a suitable position with respect to $\widetilde{L}$ according to [5] and [6].

Proposition 3.1. *Let F be a surface mentioned above. Then, we may isotope F so that;*

(i) *F meets $S_\pm$ transversely in a pairwise disjoint collection of simple closed curves and*
(ii) *F meets each crossing-ball in a collection of saddle-shaped disks* (*Figure* 4).

Figure 4. A saddle

Let F be a splitting sphere satisfying the conditions in Proposition 3.1. The *complexity* $c(F)$ of F is the lexicographically ordered pair (t, u), where t is the number of saddle-intersections of F with crossing-balls of the diagram $\widetilde{L}$, and u is the total number of components of $F \cap S_+$ and $F \cap S_-$. We say that F has *minimal complexity* if $c(F) \leq c(F')$ for any splitting sphere F'. Then, we have the following also according to [5] and [6].

Proposition 3.2. *Let F be a splitting sphere for a splittable link. If F satisfies the conditions in Proposition 3.1 and has minimal complexity, then each simple closed curve C in $F \cap S_+$* (*resp.* $F \cap S_-$) *meets the following requirements;*

(i) *C bounds a disk in F whose interior lies entirely above S_+* (*resp. below* S_-),
(ii) *C meets at least one bubble, and*
(iii) *C does not meet any bubble in more than one arc.*

We say that a splitting sphere is in *standard position* if it satifies conclusions of Proposition 3.1 and 3.2. Now let $\widetilde{L}$ be an almost alternating diagram of a splittable link L. Assume that F is a splitting sphere for L in standard position. Let A denote the crossing ball at the dealternator. Note that our almost alternating diagram has exactly one dealternator. Let C be a simple closed curve of $F \cap S_\pm$. Since $\widetilde{L}$ is almost alternating, a subarc of C satisfies the following *almost alternating property*;

If C meets two bubbles of crossing-balls B_1 and B_2 in succession. Then,

(i) *two arcs of $\widetilde{L} \cap S_\pm$ on B_1 and B_2 lie on the opposite sides of C if none of B_1 and B_2 are A and*
(ii) *two arcs of $\widetilde{L} \cap S_\pm$ on B_1 and B_2 lie on the same side of C if one of B_1 and B_2 is A.*

Moreover, C satisfies the following.

Lemma 3.3. *Every curve must pass the dealternator exactly once.*

Proof. We may assume that the complexity of F is finite. It is sufficient to show just that C passes the dealternator, since we have the third condition of Proposition 3.2. Suppose that C does not pass the dealternator. Here, we may assume that the dealternator is outside of C, that is, in the region with $\{\infty\}$ of the two regions devided by C. Note that the length of every curve is more than one and even. From the almost alternating property, C must contain at least one curve inside of it, say C'. Then, C' is also does not pass the dealternator, because it is inside of C. Since C' also must contain at least one curve inside of it, we can inductively find an infinitely many curves inside of C. This contradicts the finiteness of the complexity of F. $\square$

The collection of circles of $F \cap S_{\pm}$, together with the saddle components of $F \cap (\cup \{B_i\})$, give rise to a cell-decomposition of F, which we call the intersection graph G of F with $S_{\pm}$ and $\cup \{B_i\}$. The vertices of G correspond to the saddles of $F \cap (\cup \{B_i\})$, the edges of G correspond to the arcs of $F \cap (S_+ - \cup \{B_i\}) = F \cap (S_- \cup \{B_i\})$, and the faces of G correspond to the disks, called *dome*, bounded by the simple closed curves of $F \cap S_+$ and of $F \cap S_-$, afforded to us by the first condition of Proposition 3.2. Note that G is a plane graph in sphere F and the degree of each vertex of G is 4. We define the degree of a face as the number of vertices which the face has on its boundary. Let f_i and $\mid f_i \mid$ be a face with degree i and the number of faces of degree i, respectively. Then, we have the following from the Euler's formula.

Lemma 3.4. $\sum(i-4) \mid f_i \mid = -8$

Proof. Let n, e, and f be the numbers of the vertices, edges, and faces of G, respectively. From the Euler's formula, we have $n - e + f = 2$. Since we also have $\sum i \mid f_i \mid = 4n$, $\sum i \mid f_i \mid = 2e$ and $\sum \mid f_i \mid = f$, $\sum (i-4) \mid f_i \mid = -\sum i \mid f_i \mid + 2 \sum i \mid f_i \mid - 4 \sum \mid f_i \mid = -4n + 4e - 4f = -8$. $\square$

For a convenience, we introduce several terminologies. We call a vertex a *black vertex* and denote it by v_d if it comes from a saddle on the dealternator. And also, we denote by b_d the bubble put on the dealternator. Moreover, denote by b_i the bubble which contains a saddle corresponding to vertex v_i, which we call a *white vertex*. Put color black and white to black vertices and to other vertices, respectively. We say, as usual, a face is *adjacent* to another face if they have a common edge on their boundaries. We say that a face is *vertexwise-adjacent* to another face if they have a common vertex on their boundaries.

4. Proof of Theorem 2.3

We need the following lemma to prove Theorem 2.3. Remark here that our almost alternating diagram has exactly one dealternator.

Lemma 4.1. *Let $\widetilde{L}$ be a connected, prime, reduced almost alternating diagram of a splittable link L. If $\widetilde{L}$ is one of the diagrams in Figure 5, then there exists another connected, prime, reduced almost alternating diagram of L or of another splittable link with less crossings than $\widetilde{L}$.*

Figure 5.

Proof. We show only the case for diagram VI, which has three tangle areas T_1, T_2, and T_3 and ten regions a, b, $\ldots$, j. Some of these regions might be the same. Then we have six possibilities which regions are the same from the reducedness and the primeness of $\widetilde{L}$ (for instance, we have a nonprime

diagram if $a = d$ and we have a nonreduced diagram if $b = e$ and $c = f$). Here, we show the case that all ten regions are mutually different. Other cases can be shown similarly. In addition, let us assume that regions g and j do not share an arc inside of T_3 (in this case, we consider diagram $\widetilde{K}$ of L and diagram $\widetilde{K'}$ of another splittable link instead of $\widetilde{L}$ and $\widetilde{L}'$, see Figure 7). If link L has diagram $\widetilde{L}$, then L has diagram $\widetilde{L}'$ with one less crossings than those of $\widetilde{L}$ as well. Then, note that all nine regions k, l, ..., s are mutually different.

FIGURE 6. $\widetilde{L}$ and $\widetilde{L}'$

FIGURE 7. $\widetilde{K}$ and $\widetilde{K'}$

(Connectedness) Assume that $\widetilde{L}'$ is not connected. Then, we have a simple closed surve C in a region of $\widetilde{L}'$ such that each of the two regions of $S^2 - C$ contains a component of $\widetilde{L}'$ (here we call such a curve a *splitting curve*). If C is entirely contained in a tangle area, then it is easy to see that $\widetilde{L}$ is not connected as well. Therefore, $C \cap \{ \cup T_i \} \neq \emptyset$ and C is in region k, l, ..., or s. We may assume that C has minimal intersection with $\cup T_i$. Take a look at one of outermost intersections of T_i and C. Then the intersection is one of the following (Figure 8). In the cases of (i) and (iv), we can have another splitting curve C' which is entirely contained in T_i or which has one less intersections with $\cup T_i$ than C does. In the cases of (iii) and (vi), we have the same regions among the nine regions of $\widetilde{L}'$. In other cases, C has an intersection with $\widetilde{L}'$.

FIGURE 8.

(Primeness) Assume that $\widetilde{L}'$ is not prime. Then we have a simple closed curve C which intersects $\widetilde{L}'$ in just two points belonging to different arcs of $\widetilde{L}'$ (here we call such a curve a *separating curve*). If C is in a tangle area, then it is easy to see that $\widetilde{L}$ is not prime as well. However, $C \cap \{ \cup T_i \} \neq \emptyset$, since no pair of nine regions share two different arcs outside of tangle areas. We may assume that C has minimal intersection with $\cup T_i$. Take a look at one of the outermost intersections of T_i and C. Then the intersection is one of the figures in Figure 8. In the cases (i) and (iv), we can eliminate the intersection, which is a contradiction. In the cases of (ii) and (v), we can obtain another separating curve C' which is entirely contained in T_i or which has one less intersections with $\cup T_i$ than C does. In the cases of (iii) and (vi), we have the same regions among the nine regions of $\widetilde{L}'$. In the case of (vii), C has an intersection with T_3 and here we may assume that the other intersection is outside of T_3 from the minimality. Then regions k and n must share arcs inside and outside of T_3, and thus regions g and j must share an arc inside of T_3, which contradicts our assumption.

(Reducedness) If $\widetilde{L}'$ is diagram I, then we obtain an alternating diagram of L by reducing move I. Thus, L is non-splittable, which is a contradiction. Assume that $\widetilde{L}'$ is diagram II. Then we can find a part in the diagram which we can apply reducing move II to (Figure 9). We have four possibilities; $(x, y) = (n, o)$, (o, p), (p, q), or (q, n). In the first case, regions q and u must share a crossing so that $\widetilde{L}'$ contains the part in Figure 9. Since $m \neq o$ and $l \neq n$, we have regions t and u, and then u might be the same as m or o (see Figure 10). However, the regions which can share a crossing with region q are k, o, or s. If $u = o$, then we obtain a non-prime diagram. Therefore, this case does not occure. In the second case, regions k and q must share a crossing. Thus, we can decide the inside of T_3 more

precisely and then we can see that $\widetilde{L}$ is non-reduced (Figure 11). In the third case, regions o and v must share a crossing (Figure 12). The regions which can share a crossing with o are k, m, or q. We also have three pssibilities that $v = k$, q, or s. Thus, we obtain that $v = q$ or k. In the former case, we have a non-prime diagram. In the latter case, we can decide the inside of T_3 more precisely and then we can see that $\widetilde{L}$ is non-reduced (Figure 12). We can prove the fourth case similarly. □

FIGURE 9.

FIGURE 10.

FIGURE 11.

FIGURE 12.

Proof of Theorem 2.3. Suppose that there exists a splittable link with a connected, prime, reduced almost alternating diagram. Take all such links and consider all such diagrams of them. Let $\widetilde{L}$ be minimal in such diagrams with respect to the number of crossings. Then $\widetilde{L}$ is none of the diagrams in Figure 5, otherwise it contradicts the minimality of $\widetilde{L}$ from Lemma 4.1. Note that $\widetilde{L}$ has at least two crossings, since $\widetilde{L}$ is connected and $\widetilde{L}$ has more than one component. Let $F \subset S^3 - L$ be a splitting sphere for L, which would be assumed to be in a standard position. And let $G \subset F$ be the intersection graph of $F \cap S^2$. For each face of degree i of G, charge weight $i - 4$. We denote by $w(f)$ the weight of a face f. If there is no faces of degree 2, then every face has non negative weight. Then, $\sum (i-4) \mid f_i \mid \geq 0$ (the sum of weights of all faces), which contradicts Lemma 3.4. Therefore, we may assume that there exists at least one face of degree 2.

It may happen that two faces of degree 2 are adjacent or vertexwise-adjacent to each other. However if two faces of degree 2 are adjacent to each other, then it contradicts the reducedness of $\widetilde{L}$ (Figure 13). Also if two faces of degree 2 are vertexwise-adjacent to each other at a white vertex, then there exists a face which has two black vertices on its boundary, which contradicts Lemma 3.3. Therefore, we have two cases if we look at a face of degree 2. One is that it is not adjacent or vertexwise-adjacent to any other faces of degree 2. Here we call it a block of type T' or simply T' and then $w(T') = -2$. The other is that it is vertexwise-adjacent to another face of degree 2 at a black vertex. In this case, we put these two faces together and call it a block of type U' or simply U', and then $w(U') = -4$, which is the sum of the weights of the two faces of degree 2.

Take a look at two faces $f_{i \geq 4}$ and $f_{j \geq 4}$ which are adjacent to a block of type T' (resp. U') and put all of them together. We call it a block of type $Y_{i,j}$ (resp. $Z_{i,j}$) or simply $Y_{i,j}$ (resp. $Z_{i,j}$). In the case of $Z_{i,j}$, we assume that i is greater than equal to j. Then, $w(Y_{i,j}) = w(f_i) + w(f_j) + w(T') = i + j - 10 \geq -2$ and $w(Z_{i,j}) = w(f_i) + w(f_j) + w(U') = i + j - 12 \geq -4$. We call a face of degree i (≥ 4) a block of type X_i or simply X_i if it is not adjacent to T' or U'. Then, $w(X_i) = i - 4 \geq 0$. For blocks of type $Y_{i,j}$, the only type of blocks with a negative weight is $Y_{4,4}$ and we call a block of type $Y_{4,4}$ a block of type T or simply T. If there exists $Z_{4,4}$, then it contradicts the minimality of $\widetilde{L}$, since

Figure 13.

Figure 14. $Z_{4,4}$

we have diagram III (Figure 14). Thus, for blocks of type $Z_{i,j}$, the only type of blocks with a negative weight is $Z_{6,4}$ and we call a block of type $Z_{6,4}$ a block of type U or simply U.

If there are no blocks of type T nor type U, then it contradicts Lemma 3.4 as before. Here we say that a block is *upper* (resp. *lower*) if its faces ($\neq f_2$) come from domes which are above S_+ (resp. below S_-). Consider the following three cases; G has T_+ and no U_+ (**Case 1**), G has U_+ and no T_+ (**Case 2**), and G has T_+ and U_+ (**Case 3**), where T_+ means an upper block of type T and U_- means a lower block of type U, for instance. In each case, we show that we can discharge weights of lower blocks to T_+ and U_+ to make the weight of every block non-negative. Therefore, proving the above three cases tells us that there does not exist graph G, that is, there does not exist a connected, prime reduced almost alternating diagram of any splittable link. This completes the proof.

In each of three cases, we induce a contradiction by actually replacing the boundary cycles of subgraphs of G on the diagram and looking at the diagram as shown in Figure 13 or Figure 14. Here, put orientations on S^2 and F. We have two possibilities to replace the boundary cycle of a face on the diagram; its orientation coincides that of S^2 or not (we did not mention about this before). However, we may occasionaly choose one of the two possibilities, since the diagrams obtained by the two ways are the same up to mirror image, which does not affect our purpose. Here we have the following claim.

Claim 4.2. (i) *For every block of type T, its five white vertices come from saddles in mutually different five bubbles.*

(ii) *For blocks of type U, the boundary curves of the faces of degree four pass the same four bubbles.*

Proof. (i) Take a block of type T and put names v_α, v_β, v_γ, v_δ, and v_ε to it as shown in Figure 15. Assume that there is a pair of vertices coming from saddles in a same bubble. From the almost alternating property, we have that $b_\alpha = b_\varepsilon$, $b_\alpha = b_\delta$, or $b_\beta = b_\varepsilon$. The first case contradicts the mimimality of $\widetilde{L}$ (diagram III) and the last two cases contradict the primeness. (ii) Take two blocks of type U. Put names v_ζ, v_η, and v_θ to one of them and $v_{\zeta'}$, $v_{\eta'}$, and $v_{\theta'}$ to the other following Figure 15. Then, we have that $b_{\zeta'} = b_\theta$ and $b_{\theta'} = b_\zeta$ or that $b_{\zeta'} = b_\zeta$ and $b_{\theta'} = b_\theta$. The former case contradicts the minimality of $\widetilde{L}$ (diagram III) and the latter case contradicts the primeness unless the claim holds. □

Figure 15. T and U

Since the boundary curves of the faces of degree 2 (resp. 4) of all blocks of type $T_\pm$ (resp. $U_\pm$) pass the same two (resp. four) bubbles and are parallel (otherwise, it contradicts the primeness or the reducedness), we can define above, below, the leftside of, and the rightside of the dealternator on the diagram as shown in Figure 16. We define the top and the bottom face of T (resp. the left and the right face of U) as the face of degree 4 (resp. 2) which is above and below (resp. the leftside of and the rightside of) the dealternator on the diagram, respectively. To the boundary curves of two faces which are not vertexwise-adjacent to each other at the dealternator, we define that one is outside of the other if it is closer to the center of the dealternator than the other is on the diagram (see Figure 17). Before we start, we define the following three types of adjacency.

(A) If a face is vertexwise-adjacent to the face of degree 2 of a block of type T at the white vertex, then we say that the face is A-adjacent to the block of type T.
(B) If a face is adjacent to the top (resp. the bottom) face of a block of type T at edge $v_\delta v_\varepsilon$ (resp. $v_\alpha v_\beta$), then we say that the face is B_t- (resp. B_b-) adjacent to the block of type T.
(C) If a face is vertexwise-adjacent to the left (resp. the right) face of a block of type U, then we say that the face is C_l- (resp. C_r-) adjacent to the block of type U.

FIGURE 16.

FIGURE 17.

Case 1.

We first look at faces which are A-adjacent to blocks of type T. Since faces of degree 2 of all blocks of type T pass the same two bubbles, every face can be A-adjacent to at most one T at most once. Then we have 7 types of blocks which are A-adjacent to blocks of type T; X_i^a, $Y_{i,j}^{p,q}$, and $Z_{i,j}^{p,q}$ with $\{p, q\} = \{\cdot, a\}$, $\{a, \cdot\}$, or $\{a, a\}$, where $Z_{i,j}^{a,\cdot}$ stands for a block of type $Z_{i.j}$ whose f_i is A-adjacent to a block of type T and f_j is not, for instance. We generally use v_α, v_β, v_γ, v_δ, and v_ε to represent vertices of a block of type T as the proof of Claim 4.2 (i), which ensures us that b_α, b_β, b_γ, b_δ, and b_ε are mutually different.

Claim 4.3. (i) *No face of degree* 4 *can be adjacent to a face of degree* 2 *and* A*-adjacent to a block of type* T*.*
(ii) *No face of degree* 4 *can be adjacent to two faces of degree* 2 *with any face which is* A*-adjacent to a block of type* T*.*
(iii) *No face of degree* 6 *can be adjacent to two faces of degree* 2 *with any other face of degree* 6 *which is* A*-adjacent to a block of type* T*.*

Proof. (i) It contradicts the mimimality of $\widetilde{L}$ (diagram IV). (ii) Assume that the boundary curves of T and the face f which is A-adjacent to it have been replaced on the diagram. Put names v_1, v_2, and v_3 to the face of degree 4 as shown in Figure 18. We may assume that b_1 is below the dealternator. Here note that v_1 is on the boundary cycle of f. Therefore we have that b_1 is surrounded by boundary curve $b_\alpha b_\beta b_\gamma b_d$ or that $v_1 = v_\beta$ and $v_2 = v_\alpha$. Similarly, we have that b_3 is surrounded by boundary curve $b_\gamma b_\delta b_\varepsilon b_d$ or that $v_3 = v_\delta$ and $v_2 = v_\varepsilon$. It is easy to see that none of four cases can be held considering the length of the boundary curve of f_4. (iii) Put names $v_1, \ldots, v_8$ as shown in Figure 18. From symmetricity, we may assume that the face, say f, with $v_d v_1 v_2 v_3 v_4 v_5$ as its boundary cycle is A-adjacent to T at v_2 or v_3. Assume that the boundary curves of f and T have been replaced on the diagram. In the first case, we have that $b_6 = b_\alpha$, $b_6 = b_\beta$, $b_7 = b_\beta$, or $b_7 = b_\alpha$ considering replacing the boundary cycle $v_d v_5 v_6 v_7 v_8 v_1$ on the diagram. The first three cases contradicts the mimimality of $\widetilde{L}$ (diagram IV or V) and the last case contradicts the primeness. In the second case, we have that b_6, b_7, or $b_8 = b_\gamma$ considering replacing boundary cycle $v_d v_5 v_6 v_7 v_8 v_1$ on the diagram. The first and the third cases contradict the mimimality of $\widetilde{L}$ (diagram IV) and the second case contradicts the primeness. □

We have that $w(X_i^a) = i - 4 \geq 4 - 4 = 0$. From Claim 4.3, we obtain that $Y_{i,j}^{a,\cdot} = Y_{\geq 6, \geq 4}$, $Y_{i,j}^{\cdot,a} = Y_{\geq 4, \geq 6}$, $Y_{i,j}^{a,a} = Y_{\geq 6, \geq 6}$, $Z_{i,j}^{a,\cdot} = Z_{\geq 8, \geq 6}$, $Z_{i,j}^{\cdot,a} = Z_{\geq 8, \geq 6}$, and $Z_{i,j}^{a,a} = Z_{\geq 8, \geq 6}$. Therefore, we have that

FIGURE 18. Z_4 and $Z_{6,6}$

$w(Y_{i,j}^{a,\cdot}) = i + j - 10 \geq 0$, $w(Y_{i,j}^{\cdot,a}) \geq 0$, $w(Y_{i,j}^{a,a}) \geq 2$, $w(Z_{i,j}^{a,\cdot}) = i + j - 12 \geq 2$, $w(Z_{i,j}^{\cdot,a}) \geq 2$, and $w(Z_{i,j}^{a,a}) \geq 2$. Then, for each block which is A-adjacent to blocks of type T, discharge 2 out of its weight to each of the blocks of type T if the sum of the weights of the block and all the blocks of type T is non-negative. If the sum is negative, call it a block of type $\mathcal{A}$, $\mathcal{B}^*$, $\mathcal{C}^*$, $\mathcal{D}^*$, $\mathcal{E}^*$, $\mathcal{F}^*$, $\mathcal{G}^*$, $\mathcal{H}$, $\mathcal{I}^*$ or $\mathcal{J}^*$ as follows, where $\mathcal{B}^*$ means $\mathcal{B}$ or $\mathcal{B}'$, for instance.

The type of a block such that the sum of the weights of the block and blocks of type T which the block is A-adjacent to is negative is X_4^a, $Y_{6,4}^{a,\cdot}$, $Y_{4,6}^{\cdot,a}$, $Y_{6,6}^{a,a}$, or $Z_{8,6}^{a,a}$. We consider the first and the last three cases. In the second case, we obtain the same types as those of the third case. It is easy to see that we can uniquely obtain the diagram from X_4^a with T on the diagram and we say that the block has type $\mathcal{A}$.

Take a look at $Y_{4,6}^{\cdot,a}$ and put names $v_1, \ldots, v_7$ as shown in Figure 19. Then, its f_6 is A-adjacent to T at v_2, v_3, or v_4. In each case, replace the boundary cycle of its f_4 on the diagram assuming that we have already replaced the boundary cycles of f_6 and T. In the first case, we have two possiblities; $b_6 = b_\alpha$, b_β and $b_7 = b_\varepsilon$. In the former (resp. latter) case, we say that the block has type $\mathcal{B}$ (resp. $\mathcal{C}$) and say that a block of type $Y_{4,6}^{\cdot,a}$ has type $\mathcal{B}'$ (resp. $\mathcal{C}'$) if it represents a mirror image of the diagram for $\mathcal{B}$ (resp. $\mathcal{C}$) with T. In the second case, we also have two possiblities; $b_6 = b_\gamma$ or $b_7 = b_\gamma$, since the boundary curve of its f_4 is surrounded by the boundary curve of its f_6 on the diagram. In the fomer case, we say that the block has type $\mathcal{D}$ and define type $\mathcal{D}'$ as above. The latter case contradicts the primeness of $\widetilde{L}$. In the third case, we have two possibilities; $b_7 = b_\delta$, b_ε and $b_6 = b_\alpha$. The former case contradicts the reducedness and the latter case contradicts the minimality of $\widetilde{L}$ (diagram V).

Take a look at $Y_{6,6}^{a,a}$ and put names $v_1, \ldots, v_9$ as shown in Figure 19. Call the face with v_1 (resp. v_9) a face f (resp. f'). Let T_1 and T_2 be two blocks of type T. Let f (resp. f') be A-adjacent to T_1 (resp. T_2). Here we assume that T_1 (resp. T_2) has vertices v_α, v_β, v_γ, v_δ, and v_ε (resp. $v_{\alpha'}$, $v_{\beta'}$, $v_{\gamma'}$, $v_{\delta'}$, and $v_{\varepsilon'}$). From the symmetricity, we may assume that the boundary curve of f passes the rightside of the dealternator on the diagram and then, f is A-adjacent to T_1 at v_3. Replace the boundary cycles of f and T_1 on the diagram. Now we have two possibilities to replace the boundary cycle of f'; $b_6 = b_\gamma$ or $b_8 = b_\gamma$ from the almost alternating property. In the first case, we have that $b_7 = b_{\beta'}$. If $b_{\alpha'} = b_\delta$ or $b_{\beta'} = b_1$, then it contradicts the reducedness. Therefore, we have that $b_{\alpha'} = b_1$ or $b_{\beta'} = b_\delta$. In the former (resp. latter) case, we say that the block has type $\mathcal{E}$ (resp. $\mathcal{F}$) and define type $\mathcal{E}'$ and type $\mathcal{F}'$ as before. In the second case, we have that $b_7 = b_{\delta'}$ and $b_9 = b_{\beta'}$. If $b_{\alpha'} = b_\delta$, then it contradicts the minimality of $\widetilde{L}$ (diagram IV). If $b_{\beta'} = b_\delta$, then it contradicts the primeness. Thus, we may assume that $b_{\alpha'} = b_1$ and that $b_{\delta'} = b_\beta$ or $b_{\varepsilon'} = b_\beta$. If $b_{\varepsilon'} = b_\beta$, then it contradicts the minimality, again. Therefore, we have that $b_{\varepsilon'} = b_\alpha$ and $b_{\delta'} = b_\beta$, and then we say that the block has type $\mathcal{G}$ and define type $\mathcal{G}'$ as above.

At last, take a look at $Z_{8,6}^{a,a}$ and put names $v_1, \ldots, v_{10}$ as shown in Figure 19. Let its f_6 and its f_8 be A-adjacent to, a block of type T, T_1 and T_2, respectively. Replace the boundary cycle of its f_8 on the diagram assuming that we have already replaced the boundary cycles of the f_6 and T_1. First, assume that the boundary curve of the f_6 passes the rightside of the dealternator on the diagram, and then it is A-adjacent to T_1 at v_9. Then, its f_8 must be A-adjacent to T_2 at v_4 from the almost alternating property and the minimality of $\widetilde{L}$ (diagram IV). Then, we have that $b_3 = b_{\delta'}$ and $b_5 = b_{\beta'}$. If $b_{\alpha'} = b_\delta$ or $b_{\varepsilon'} = b_\beta$, then it contradicts the minimality of $\widetilde{L}$ (diagram IV). Thus, we have that $b_{\beta'} = b_\delta$ and $b_{\delta'} = b_\beta$. Then, we say that the block has type $\mathcal{H}$. Second, assume that the boundary curve of the f_6 passes the leftside of the dealternator on the diagram, and then we may assume that it is A-adjacent to T_1 at v_{10} from the symmetricity of the block. Then, its f_8 is A-adjacent to T_2 at v_3

or v_5 from the almost alternating property. In the first case, it contradicts the reducedness if $b_4 = b_\varepsilon$ and the primeness if $b_5 = b_\varepsilon$. It also contradicts the the minimality of $\widetilde{L}$ (diagram IV) if $b_6 = b_\delta$ or $b_6 = b_\varepsilon$. Therefore, we have that $b_4 = b_\delta$ or $b_5 = b_\delta$. In the former (resp. latter) case, we call the block has type $\mathcal{I}$ (resp. $\mathcal{J}$). In the second case, we have that $b_6 = b_\delta$ or $b_6 = b_\varepsilon$. Both cases contradict the minimality of $\widetilde{L}$ (diagram IV).

FIGURE 19. $Y_{4,6}$, $Y_{6,6}$, and $Z_{8,6}$

FIGURE 20.

Claim 4.4. *Let $L = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$, $L' = \{\mathcal{B}', \mathcal{C}'\}$, $M = \{\mathcal{G}, \mathcal{H}\}$, $M' = \{\mathcal{G}'\}$, $N = \{\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{I}, \mathcal{J}\}$, and $N' = \{\mathcal{D}', \mathcal{E}', \mathcal{F}', \mathcal{I}', \mathcal{J}'\}$. Then, we have the following.*

(i) *A block of any type of $L \cup L' \cup M \cup M'$ does not coexist in graph G with any other block of a different type. Blocks of the same type of $L \cup L' \cup M \cup M'$ can coexist in graph G. Then, their boundary curves of their top (resp. bottom) faces pass the same four bubbles.*

(ii) *Any block of N does not coexist in the graph G with any blocks of N'. Blocks of N (resp. N') may coexist in graph G. Then, the boundary curves below (resp. above) the dealternator pass the same five bubbles.*

Proof. (i) We say a subgraph of G a subblock of type P, Q, R, and S (resp. P', Q', R', and S') if its boundary curve constracts a diagram P, Q, R, and S (resp. the mirror image P', Q', R', and S') of Figure 21, respectively. It is easy to see that it contradicts the mimimality of $\widetilde{L}$ (diagram IV) if graph G has S (resp. S') and P', Q' and R' (resp. P, Q and R). Here note that any block of a type of $L \cup L' \cup M \cup M'$ consists of one of P, Q and R and one of P', Q' and R' (for instance, a block of type $\mathcal{G}$ consists of a subblock of type P and a subblock of type R'). In addition, any block of N (resp. N') contains a subblock of type S' (resp. S). Therefore, any block of $L \cup L' \cup M \cup M'$ and any block of $N \cup N'$ cannot coexist in graph G. From the primeness, the type of a block whose boundary curve can exist inside of the boundary curve of P is only P among P, Q, and R, and then their boundary curves pass the same bubbles. Next, assume that there is the boundary curve $b_d b_\gamma b_\delta b_\varepsilon$ of a face of degree 4 inside of a boundary curve $b_d b_{\gamma'} b_{\delta'} b_{\varepsilon'}$ of a subblock of type Q. Then we have that $b_\delta = b_{\delta'}$ and $b_\varepsilon = b_{\varepsilon'}$, $b_\delta = b_1$ and $b_\varepsilon = b_{\varepsilon'}$, $b_\delta = b_2$, or $b_\varepsilon = b_2$. The last three cases contradicts the primeness. Therefore consider the first case. If the face is of a subblock of type P, then it contradicts primeness. If the face is of a subblock of type Q, then their boundary curves pass the same 6 bubbles from the primeness. If the face is of a subblock of type R, then we cannot connect bubbles b_4 and b_δ with an arc for R (see Figure 21). Now assume that there is the boundary curve $b_d b_\gamma b_\delta b_\varepsilon$ of a face of degree 4 inside of the boundary curve $b_d b_{\gamma''} b_{\delta''} b_{\varepsilon''}$ of a subblock of type R. Then we have that $b_\delta = b_{\delta''}$, $b_\varepsilon = b_{\delta''}$, $b_\delta = b_4$, or $b_\varepsilon = b_4$. The last two cases contradicts the primeness. Consider the first case. Then, we also obtain that $b_\varepsilon = b_{\varepsilon''}$ from the primeness. If the face is of a subblock of type P or Q, then it contradicts the primeness. If the face is of a subblock of type R, then their boundary curves pass the same 6 bubbles also from the primeness. Next, consider the second case. Note that we are now considering the coexistence of blocks of $L \cup L' \cup M \cup M'$. Therefore, we have a subblock of type P', Q', or R'. Then it contradicts the minimality of $\widetilde{L}$ (diagram IV). Now we need to show that a block of type $\mathcal{C}$ and a block of type $\mathcal{G}$ (or $\mathcal{C}'$ and $\mathcal{G}'$) do not coexist in graph G. If graph G has $\mathcal{C}$, then the boundary curve of the top and bottom face of any block of type T must pass the same 4 bubbles as the boundary curve of the top and bottom face of the block of type $\mathcal{C}$, respectively. However, $\mathcal{G}$ has two top faces whose boundary curves do not pass the same bubbles. It is a contradiction. (ii) If any

of N and any of N' coexist in graph G, then we have S and S' on the diagram, which contradicts the reducedness. It is easy to see the last part following the previous case. $\square$

FIGURE 21.

We devide Case 1 into the following 6 subcases; case 1-$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{G}$, $\mathcal{H}$, and N according that there is a block of $\mathcal{A}$, $\mathcal{B}^*$, $\mathcal{C}^*$, $\mathcal{G}^*$, $\mathcal{H}^*$, and N, respectively. In each case, we look at the block which is B_*-adjacent to a block of type T with a negative weight. Then, We have the following.

Claim 4.5. (i) *No face of degree* 4 *can be* B_b*- (or* B_t*-) adjacent to a block of type* T*.*
(ii) *No face of degree* 6 *can be* B_b*- and* B_t*-adjacent to a block or blocks of type* T*.*
(iii) *No face of degree* 8 *can be* A*-,* B_b*-, and* B_t*-adjacent to blocks of type* T*.*

Proof. (i) Assume that we have a face of degree 4 with boundary cycle $v_\alpha v_\beta v_1 v_2$. Considering the length of the cycle, we have that $v_1 = v_d$ and $b_2 = b_\delta$ or b_ε. The former case contradicts the reducedness and the latter case contradicts the mimimality of $\widetilde{L}$ (diagram V). (ii) Assume that we have a face of degree 6 which is B_b- and B_t-adjacent to a block of type T and its boundary cycle is $v_\alpha v_\beta v_1 v_2 v_3 v_4$ (it can be similarly shown the case that the face is B_b- and B_t-adjacent to blocks of type T). Considering the length of the cycle and the almost alternating property, we have that $v_1 = v_d$, $b_2 = b_\delta$, and $b_3 = b_\varepsilon$. Then, it contradicts the reducedness. (iii) We show only the case that we have a face of degree 8 which is A-adjacent to a block of type T and B_b-, and B_t-adjacent to another block of type T. And let $v_\alpha v_\beta v_1 v_2 v_3 v_4 v_5 v_6$ be the boundary cycle of the face. Considering the length of the cycle and the almost alternating property, we have that the face is B_t-adjacent to the block at $v_2 v_3$ or $v_4 v_5$. The former case contradicts the reducedness. In the latter case, the face must be A-adjacent to the block at v_6, and then it contradicts the reducedness, again. $\square$

Case 1-$\mathcal{A}$. Take a block of type $\mathcal{A}$ and a block of type T which are A-adjacent to each other and put them together. We call it a block of type $T_\mathcal{A}$ or simply $T_\mathcal{A}$, and so $w(T_\mathcal{A}) = -2$. Note that we do not have any of $\{L \cup L' \cup M \cup M' \cup N \cup N'\} - \{\mathcal{A}\}$ from Claim 4.4. Take a look at blocks which are B_*-adjacent to blocks of type $T_\mathcal{A}$. We define, for instance, $Z_{i,j}^{a*,bt}$ as a block whose f_i is A- and B_*-adjacent to blocks of type $T_\mathcal{A}$ and whose f_j is B_b- and B_t-adjacent to blocks of type $T_\mathcal{A}$. Also, we use Z_i^a for a face f_i which is adjacent to two faces of degree 2 and is A-adjacent to $T_\mathcal{A}$.

Claim 4.6. *Graph* G *does not have any block of type* X_6^{a*}, X_{10}^{abt}, $Z_4^{\cdot}$, Z_6^a, Z_8^{a*}, $Y_{8,j}^{bt,a}$, *or* $Z_{6,j}^{*,a}$.

Proof. Let $v_\alpha v_\beta v_1 v_2 v_3 v_4$ be the boundary cycle of X_6^{ab}. Note that the boundary curve must pass the dealternator and curve $b_d b_\gamma b_\delta b_\varepsilon$. Thus, X_6^{ab} must be A-adjacent to a block of type T at v_4 from the almost alternating property. However then, it contradicts the minimality of $\widetilde{L}$ (diagram IV). We can similarly show that there does not exist X_{10}^{abt}.

Let $v_d v_1 v_2 v_3$ be the boundary cycle of $Z_4^{\cdot}$ and let b_1 (resp. b_3) be inside of $b_d b_\alpha b_\beta b_\gamma$ (resp. $b_d b_\gamma b_\delta b_\varepsilon$). Since the length of curve $b_d b_1 b_2 b_3$ is 4 and it must pass $b_d b_\alpha b_\beta b_\gamma$ and $b_d b_\gamma b_\delta b_\varepsilon$, we have that $b_1 = b_\alpha$, b_β or that $b_3 = b_\delta$, b_ε. However if $b_1 = b_\alpha$ or $b_3 = b_\varepsilon$, then it contradicts the primeness. And if $b_1 = b_\beta$ or $b_3 = b_\delta$, then it contradicts the minimality of $\widetilde{L}$ (diagram IV). In the case of Z_6^a, let $v_d v_1 v_2 v_3 v_4 v_5$ be its boundary cycle. Following the previous argument, we have that $b_1 \neq b_\alpha$, b_β and $b_5 \neq b_\delta$, b_ε. From the length of the cycle, Z_6^a must be A-adjacent to a block of type T at v_3. Then, we have that $b_2 = b_\alpha$ or b_β. The first case contradicts the minimality of $\widetilde{L}$ (diagram IV) and the second case contradicts the primeness. We can similarly show that there does not exist Z_8^{a*}.

Let $v_\alpha v_\beta v_1 v_2 v_3 v_4 v_5 v_6$ be the boundary cycle of Y_8^{bt}. It is B_t-adjacent to T at $v_2 v_3$ or $v_4 v_5$ from the almost alternating property. The former case contradicts the reducedness. In the latter case, we have that $b_3 = b_d$ and $b_6 \neq b_\gamma$ from the reducedness and the minimality of $\widetilde{L}$ (diagram IV). Therefore, b_γ and the boundary curve of f_j of $Y_{8,j}^{bt,a}$ is in one and in the other of the two regions of $S^2 - b_\alpha b_\beta b_1 b_2 b_3 b_4 b_5 b_6$, respectively. Therefore, f_j cannot be A-adjacent to T. We can similarly show that there does not exist $Z_{6,j}^{*,a}$. □

Claim 4.7. *The boundary curve of the face ($\neq f_2$) of any of X_8^{a*}, X_8^{bt}, Y_6^a, and Z_6^* passes the leftside of the dealternator on the diagram.*

Proof. We show the proof only for X_8^{ab} and Y_6^a. Let $v_\alpha v_\beta v_1 v_2 v_3 v_4 v_5 v_6$ be the boundary cycle of X_8^{ab}. Note that the boundary curve must pass the dealternator and curve $b_d b_\gamma b_\delta b_\varepsilon$. Thus, X_8^{ab} must be A-adjacent to a block of type T at v_4 or v_6 from the almost alternating property. Then we have the former case and that $v_2 = v_d$ from the minimality of $\widetilde{L}$ (diagram IV), which completes the proof. Moreover, we have that $b_3 = b_\delta$ from the primeness and then we obtain the diagram shown in Figure 22.

Let $v_{\gamma'} v_1 v_2 v_3 v_4 v_5$ be the boundary cycle of Y_6^a. Since the length of the boundary curve is 6 and it must pass the boundary curves of the top and bottom faces of the block of type T of $T_{\mathcal{A}}$, we have that $b_d = b_2$, b_3, or b_4. If $b_d = b_3$, then we have that $b_1 = b_\alpha$, b_β or that $b_2 = b_\alpha$, b_β. The first and fourth cases contradict the mimimality of $\widetilde{L}$ (diagram IV) and the second case contradicts the primeness. Therefore we have that $b_2 = b_\alpha$ and then, we similarly obtain that $b_4 = b_\varepsilon$. However, then it contradicts the primeness, since Y_6^a is adjacent to a face of degree 2 at $v_2 v_3$ or at $v_3 v_4$. This completes the proof. Moreover, if $b_d = b_2$, then $b_1 = b_\alpha$ or b_β. The former case contradicts the mimimality of $\widetilde{L}$ (diagram IV). In the latter case, Y_6^a must be adjacent to a face of degree 2 at $v_2 v_3$ from the minimality of $\widetilde{L}$ (diagram IV). Then, we have that $b_3 = b_\delta$, b_ε, that $b_4 = b_\delta$, b_ε, or that $b_5 = b_\delta$, b_ε. All the five cases but the third contradict the primeness or the minimality of $\widetilde{L}$ (diagram IV). Therefore, we obtain the diagram shown in Figure 22. In the case that $b_4 = b_d$, we obtain a mirror image of the diagram. □

FIGURE 22. X_8^{a*} and Y_6^a with $T_{\mathcal{A}}$ and T'

Claim 4.7 says that there does not exist any $X_{i,j}$ or $Y_{i,j}$ such that each of its faces f_i and f_j is X_8^{a*}, X_8^{bt}, Y_6^a, or Z_6^*. Moreover, we have the following.

Claim 4.8. *Graph G does not have any block of type $Y_{4,8}^{\cdot,a*}$, $Y_{6,6}^{a,*}$, or $Y_{6,8}^{*,a*}$.*

Proof. Take a look at the diagram of X_8^{a*} with $T_{\mathcal{A}}$ and T' (Figure 22). From the mimimality of $\widetilde{L}$ (diagram IV), Y_8^{a*} must be adjacent to a face of degree 2 at $b_1 b_2$. Then, it is easy to see that there does not exist $Y_{4,8}^{\cdot,a*}$, since the length of the boundary curve of $Y_4^{\cdot}$ is 4 and it must pass $b_d b_\alpha b_\beta b_\gamma$, $b_d b_{\alpha'} b_{\beta'} b_{\gamma'}$, and $b_d b_\gamma b_\delta b_\varepsilon$. We can similarly show that there does not exist $Y_{6,6}^{a,*}$ or $Y_{6,8}^{*,a*}$. □

We are now looking at blocks which are B_*-adjacent to blocks of type $T_{\mathcal{A}}$. Each face ($\neq f_2$) of every such a block can be B_b- (resp. B_t-) adjacent to at most one $T_{\mathcal{A}}$ at most once from Claim 4.4 (i). In addition, the face might be A-adjacent to a block of type T as well, and then note that we have discharged weight 2 of the face to the block of type T. Then we have 68 types of blocks which are B_*-adjacent to blocks of type $T_{\mathcal{A}}$; X_i^p, $Y_{i,j}^{q,r}$, and $Z_{i,j}^{q,r}$ with $\{q, r\} = \{\cdot, *, bt, a, a*, abt\}$ and p and one of q and r are of $\{*, bt, a*, abt\}$. Consider the sum of weights of such a block and blocks of type T or

type $T_{\mathcal{A}}$ which it is A- or B_*-adjacent to. Then the type of a block such that the sum is negative is $Y^{\cdot,*}_{4,6}$, $Y^{*,\cdot}_{6,4}$, $Z^{\cdot,*}_{6,6}$, $Y^{*,\cdot}_{6,4}$, $Y^{*,*}_{6,6}$, $Z^{*,*}_{8,6}$, $Z^{bt,bt}_{10,8}$, $Y^{\cdot,bt}_{4,8}$, $Y^{bt,\cdot}_{8,4}$, $Z^{bt,\cdot}_{8,6}$, $Y^{*,bt}_{6,8}$, $Y^{bt,*}_{8,6}$, $Z^{*,bt}_{8,8}$, $Z^{bt,*}_{8,8}$, or $Z^{bt,*}_{10,6}$ from Claim 4.3, 4.5, 4.6, 4.7, and 4.8 and then the sum is -2. Next we consider such blocks.

In each case of $Y^{b,b}_{6,6}$ and $Y^{t,t}_{6,6}$, choose one of two blocks of type $T_{\mathcal{A}}$ and discharge its weight 2 to the block. In the case of $Z^{*,*}_{8,6}$, discharge the weight 2 to the block of type $T_{\mathcal{A}}$ which its f_8 is B_*-adjacent to. Now, in each of the above 3 cases and the first 4 of 16 cases, we have the situation that a block with its weight 0 is B_*-adjacent to $T_{\mathcal{A}}$ with its weight -2. We say that such blocks are type I. In each of the last 5 of 16 cases, discharge 2 out of its weight 4 to the block of type $T_{\mathcal{A}}$ which it is B_*-adjacent to. In the case of $Z^{bt,bt}_{10,8}$, discharge 2 out of its weight 6 to each of the two blocks of type $T_{\mathcal{A}}$ which its f_{10} is B_b- and B_t-adjacent to. Now, in each of the above 6 cases and the cases of $Y^{\cdot,bt}_{4,8}$, $Y^{bt,\cdot}_{8,4}$, $Y^{b,t}_{6,6}$, $Y^{t,b}_{6,6}$, and $Z^{bt,\cdot}_{8,6}$, we have that a block with its weight 2 is B_b- and B_t-adjacent to two blocks of type $T_{\mathcal{A}}$ with each weight -2 (if the two blocks of type $T_{\mathcal{A}}$ are the same, we can discharge the weight 2 to the block of type $T_{\mathcal{A}}$ and make its weight non-negative. Therefore, we do not consider such a case). We say that such blocks are type II.

Then, we can constract paths by regarding blocks of type T with negative weights and blocks of type I and type II as edges and vertices, respectively. Here, note that each block of type I is B_*-adjacent to exactly one T with a negative weight and each block of type II is B_b- and B_t-adjacent to exactly two blocks of type T with negative weights. Therefore, for each path, if the block corresponding to one of its ends is B_b-adjacent to T with a negative weight, then the block corresponding to the other of its ends is B_t-adjacent to T with a negative weight. Now we have the following.

Claim 4.9. *Assume that diagram D contains at least one from each of $\{P, Q, R\}$. Let χ (resp. χ') be an arc $b_1b_2b_3b_4$ such that $b_1 = b_\beta$, $b_2 = b_\alpha$, and $b_4 = b_\delta$ (resp. $b_1 = b_\delta$, $b_2 = b_\varepsilon$, and $b_4 = b_\beta$). Let ψ (resp. ψ') be an arc $b_1b_2b_3$ such that $b_1 = b_\beta$, $b_2 = b_\alpha$, and $b_3 = b_\delta$ (resp. $b_1 = b_\delta$, $b_2 = b_\varepsilon$, and $b_3 = b_\beta$). Let ζ (resp. ζ') be an arc b_1b_2 such that $b_1 = b_\beta$ and $b_2 = b_\delta$ (resp. $b_1 = b_\delta$ and $b_2 = b_\beta$) and it sees the center of b_β on the same (resp. opposite) side as it does b_γ (Figure 23). Suppose that D contains one of $\{\chi, \psi, \zeta\}$. Then, D contains none of $\{\chi', \psi', \zeta'\}$.*

Proof. Assume that D contains χ. Then, any arc passing b_β and b_ε must passes b_α. Thus, D does not contain χ' or ψ'. Since ζ' passes b_β (resp. b_δ) seeing it at the opposite (resp. same) side as b_γ, it must be surrounded by (resp. it must surround) the cycle containing χ. It is a contradiction. Thus D does not contain ζ'. Assume that D contains ψ and ζ'. Then the diagram contains S'. It contradicts the mimimality of $\widetilde{L}$ (diagram IV), since D contains P, Q, or R. We can similarly show other cases. □

FIGURE 23. χ, ψ, and ζ

Then, the following claim says that there does not eixst such a path.

Claim 4.10. *Assume that there exist a block of type $T_{\mathcal{A}}$ and a block of type I and they are B_b- (resp. B_t-) adjacent to each other. Then, there are no other blocks of type $T_{\mathcal{A}}$ and no blocks of type I such that they are B_t- (resp. B_b-) adjacent to each other.*

Proof. From Claim 4.9, it is sufficient to show that we have an arc χ, ψ, or ζ on the diagram under the assumptionthat there exists $Y^{\cdot,b}_{4,6}$, $Y^{b,\cdot}_{6,4}$, $Y^{b,b}_{6,6}$, or Z^{b}_{6}. Let f_6 be B_b-adjacent to $T_{\mathcal{A}}$. Then, its boundary curve $b_\alpha b_\beta b_1 b_2 b_3 b_4$ must pass b_δ or b_ε. In the former case, we have an arc χ or ψ from the mimimality of $\widetilde{L}$ (diagram IV). In the latter case, we have that $b_3 = b_d$ and $b_4 = b_\varepsilon$ from the mimimality of $\widetilde{L}$ (diagram IV) and the primeness. Moreover, it can be adjacent to at most one f_2 at v_2v_3 from the

primeness. Thus, we do not have Z_6^b in this case. If we have $Y_{4,6}^{\cdot,b}$ or $Y_{6,4}^{b,\cdot}$, then the boundary curve of f_4 must pass b_β and b_δ, and thus we have ζ. If we have $Y_{6,6}^{b,b}$, then the boundary curve of another f_6 must pass b_β, b_α, and b_δ from the primeness. Then, we have χ or ψ. □

Case 1-$\mathcal{B}$. We show only the case that we have a block of type $\mathcal{B}$. The case that we have a block of type $\mathcal{B}'$ can be shown similarly. Define a block of type $T_{\mathcal{B}}$ following Case 1-$\mathcal{A}$ and take a look at blocks which are B_*-adjacent to $T_{\mathcal{B}}$. It is easy to see that the boundary curves of the top (resp. bottom) faces of any blocks of type T pass the same four bubbles as that of the top (resp. bottom) face of T of a block of type $T_{\mathcal{B}}$. This induces that the boundary curves of the top (resp. bottom) faces of T of all blocks of type $T_{\mathcal{B}}$ pass the same four babbles. Therefore, any face which is A-adjacent to T cannot be B_*-adjacent to $T_{\mathcal{B}}$. Moreover, any face which is B_*-adjacent to $T_{\mathcal{B}}$ must be adjacent to at least one f_2, since its boundary curve is outside of the boundary curve of the face of degree 6 of $T_{\mathcal{B}}$. Therefore, we need to take a look at the blocks of type $Y_{i,j}^{p,q}$ or $Z_{i,j}^{p,q}$, where p, $q \in \{\ \cdot, a, *, bt\}$ and one of them is of $\{*, bt\}$.

From Claim 4.3 and Claim 4.5, we know that G does not have X_4^*, X_6^{bt}, or Y_4^a. We can similarly show that G does not have $Z_4^{\cdot}$ or Z_6^a. Moreover, by showing that the boundary curve of Z_6^* and Z_8^{bt} passes the leftside of the dealternator on the diagram, respectively, we also obtain that G does not have any block of type $Z_{6,6}^{*,*}$, $Z_{8,6}^{bt,*}$, or $Z_{8,8}^{bt,bt}$. Then the type of a block such that the sum of its weight and weights of blocks of type $T_{\mathcal{B}}$ which it is B_*-adjacent to is negative is $Y_{6,4}^{*,\cdot}$, $Y_{4,6}^{\cdot,*}$, $Z_{6,6}^{*,\cdot}$, $Z_{6,6}^{\cdot,*}$, $Y_{6,6}^{*,a}$, $Y_{6,6}^{a,*}$, $Z_{8,6}^{a,*}$, $Y_{8,4}^{bt,\cdot}$, $Y_{4,8}^{\cdot,bt}$, $Y_{8,6}^{bt,a}$, $Y_{6,8}^{a,bt}$, $Z_{8,6}^{bt,\cdot}$, $Z_{8,8}^{bt,a}$, $Z_{8,8}^{a,bt}$, $Y_{6,6}^{*,*}$, $Z_{8,6}^{*,*}$, $Z_{10,6}^{bt,*}$, $Y_{8,8}^{bt,bt}$, $Z_{10,8}^{bt,bt}$, $Y_{8,6}^{bt,*}$, $Y_{6,8}^{*,bt}$, $Z_{8,8}^{bt,*}$, or $Z_{8,8}^{*,bt}$, and then the sum is -2.

In the case of $Y_{6,6}^{*,*}$, choose one of the two blocks of type $T_{\mathcal{B}}$ which it is B_*-adjacent to and discharge the weight 2 of $Y_{6,6}$ to the block of type $T_{\mathcal{B}}$. In the case of $Z_{8,6}^{*,*}$ (resp. $Z_{10,6}^{bt,*}$), discharge 2 out of its weight to each of the block of type $T_{\mathcal{B}}$ which its f_8 (resp. f_{10}) is B_*-adjacent to. Now, in each of the above 3 cases and the first 7 of 23 cases, we have the situation that a block with its weight 0 is B_*-adjacent to $T_{\mathcal{B}}$. We call that such blocks are type I as before. In each of the last 4 of 23 cases, take a look at the face which is B_*-adjacent to only one $T_{\mathcal{B}}$ and discharge 2 out of its weight 4 to the block of type $T_{\mathcal{B}}$. In the case of $Y_{8,8}^{bt,bt}$, choose one of two faces of degree 8 and discharge 2 out of its weight 6 to each of the blocks of type $T_{\mathcal{B}}$ which the face is B_b- or B_t-adjacent to. In the case of $Z_{10,8}^{bt,bt}$, discharge 2 out of its weight 6 to each of the blocks of type $T_{\mathcal{B}}$ which its f_{10} is B_b- or B_t-adjacent to. Now, in each of the above 4 cases and the second 7 of 23 cases, we have the situation that a block with its weight 2 is B_*-adjacent to two blocks of type $T_{\mathcal{B}}$ with each weight -2 (if the two blocks are the same, we can discharge the weight 2 to the block of type $T_{\mathcal{B}}$ and make its weight non-negative. Therefore, we do not think about such a case). We say that such blocks are type II as before.

Then we can constract paths as we did in Case 1-$\mathcal{A}$. Note that Claim 4.9 holds in this case as well. Moreover, for each block of type I, we can find an arc χ or ψ (resp. ψ') in the boundary curve of f_6 which is B_b- (resp. B_t-) adjacent to $T_{\mathcal{B}}$ from the primeness. Thus a similar claim to Claim 4.10 holds, which tells us that there does not exist such a path.

Case 1-$\mathcal{C}$. We show only the case that we have a block of type $\mathcal{C}$ as the previous case. Define a block of type $T_{\mathcal{C}}$ as before and take a look at the face f which is B_t-adjacent to a block of type $T_{\mathcal{C}}$. Similarly to Case 1-$\mathcal{B}$, we have that f cannot be A-adjacent to any block of type T and that f must be adjacent to at least one f_2. Therefore, we need to take a look at the blocks of type $Y_{i,j}^{p,q}$ or $Z_{i,j}^{p,q}$, where p, $q \in \{\cdot$, t, $a\}$ and one of them is t. It is easy to check that graph G does not have Y_6^t, $Z_4^{\cdot}$, or Z_6^a. Additionally using Claim 4.3 and Claim 4.5, we can see that the sum of the weights of such a block and the block of

type $T_{\mathcal{C}}$ which the block is B_t-adjacent to is non-negative for each case. Therefore, we can discharge the weights of such blocks to blocks of type $T_{\mathcal{C}}$ to make the weight of every block non-negative.

Case 1-$\mathcal{G}$. We show only the case that we have a block of type $\mathcal{G}$ as the previous case. Discharge the weight 2 of the block to the inner block of type T which $\mathcal{G}$ is A-adjacent to (the one whose boundary curve is surrounded by the boundary curve of the other on the diagram). Take a block of type $\mathcal{G}$ and the outer block of type T and put them together. We call it a block of type $T_{\mathcal{G}}$ or simply $T_{\mathcal{G}}$, and so $w(T_{\mathcal{G}}) = -2$. Take a look at the face f which is B_t-adjacent to a block of type $T_{\mathcal{G}}$. Then, it cannot be adjacent to any face of degree 2, since its boundary curve is inside of the boundary curve of the face which is adjacent to the inner block of type T at v_d. Therefore, we need to take a look at the blocks of type X_i^p, where p is t or at. It is easy to check that graph G does not have X_6^{at}. Additionally using Claim 4.5, we can see that the sum of the weights of such a block and the block of type $T_{\mathcal{G}}$ which the block is B_t-adjacent to is non-negative for each case. Therefore, we can discharge the weights of such blocks to blocks of type $T_{\mathcal{G}}$ to make the weight of every block non-negative.

Case 1-$\mathcal{H}$. We assume that we have a block of type $\mathcal{H}$. Define $T_{\mathcal{H}}$ as we did in Case 1-$\mathcal{G}$. Take a look at a face f which is B_t-adjacent to $T_{\mathcal{H}}$. If f is a face of a block $Y_{i,j}$ or $Z_{i,j}$, then let f' be the other face of the block which is adjacent to a face of degree 2 with f. Note that f' cannot be B_*-adjacent to any blocks of type $T_{\mathcal{H}}$, since its boundary curve is inside of the boundary curve of the face of degee 8 of $\mathcal{H}$. It is easy to see that f is not a face of type X_4^t or X_6^{at} and that f' is not a face of degree 4 or a face of type Y_6^a. Moreover, we can easily obtain that graph G does not have Z_6^t or Z_8^{at}. Therefore, we can see that the sum of the weights of such a block and the block of type $T_{\mathcal{H}}$ which the block is B_t-adjacent to is non-negative for blocks of type X_i^p, $Y_{i,j}^{q,r}$, and $Z_{i,j}^{q,r}$, where p and one of q, r are of $\{t, at\}$ and the other is of $\{\cdot, a\}$. Therefore, we can discharge the weights of such blocks to blocks of type $T_{\mathcal{H}}$ to make the weight of every block non-negative.

Case 1-N. Assume that we have at least one block of a type of N. We can similarly show the case that we have a block of a type of N'. We show this case step by step.

Step 1: To each of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{I}$, and $\mathcal{J}$, discharge its weight 2 to the inner block of type T which it is A-adjacent to. Define $T_{\mathcal{D}}$ as the union of T and $\mathcal{D}$. Define $T_{\mathcal{E}}$, $T_{\mathcal{F}}$, $T_{\mathcal{I}}$, and $T_{\mathcal{J}}$ as the union of the outer T and $\mathcal{E}$, $\mathcal{F}$, $\mathcal{I}$, and $\mathcal{J}$, respectively. Call the block which is B_b-adjacent to T_k a block of type f_k, where k is $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{I}$, or $\mathcal{J}$. No face can be B_b-adjacent to more than one such a block of type T from Claim 4.4 (ii). In addition, such a face can be adjacent to at most one face of degree 2. Therefore, we may assume that the type of a block which has such faces is X_i^p or $Y_{i,j}^{q,r}$, where one of q, r is $\cdot$ or a and the other and p are b or ab. If the boundary curve of a block of type T forms a diagram shown in Figure 24 with boundary curves of some faces which it is adjacent to, then we say that the block of type T is also type Γ or type Δ. Then, we have the following.

Claim 4.11. (i) *If $w(f_{\mathcal{D}}) + w(T_{\mathcal{D}}) < 0$, then $f_{\mathcal{D}}$ is type X_6^{ab} or $Y_{8,4}^{ab,\cdot}$ and the block of type T which it is A-adjacent to is type Γ or Δ.*

(ii) *If $w(f_{\mathcal{E}}) + w(T_{\mathcal{E}}) < 0$, then $f_{\mathcal{E}}$ is type X_6^{ab} and the block of type T which it is A-adjacent to is type Γ or Δ.*

(iii) *If $w(f_{\mathcal{F}}) + w(T_{\mathcal{F}}) < 0$, then $f_{\mathcal{F}}$ is type X_6^{ab} and the block of type T which it is A-adjacent to is type Γ.*

(iv) *If $w(f_{\mathcal{I}}) + w(T_{\mathcal{I}}) < 0$, then $f_{\mathcal{I}}$ is type X_6^{ab} and the block of type T which it is A-adjacent to is type Γ.*

(v) *$w(f_{\mathcal{J}}) + w(T_{\mathcal{J}}) \geq 0$.*

Proof. (i) It is easy to see that $f_{\mathcal{D}}$ is X_i^b, X_i^{ab}, $Y_{i,4}^{b,\cdot}$, or $Y_{i,4}^{ab,\cdot}$. If $X_i^b = X_{\geq 6}$, $X_i^{ab} = X_{\geq 8}$, $Y_{i,4}^{b,\cdot} = Y_{\geq 8,4}$, $Y_{i,4}^{ab,\cdot} = Y_{\geq 10,4}$, then $w(f_{\mathcal{D}}) + w(T_{\mathcal{D}}) \geq 0$. From Claim 4.5, we do not have X_4^b. Let $v_d v_1 v_2 v_3 v_4 v_5$ be the boundary cycle of Y_6^b. Then, the face is adjacent to f_2 at $v_d v_1$. From the almost alternating property,

it must be B_b-adjacent to $T_{\mathcal{D}}$ at v_3v_4. Then, we have that $b_5 = b_\delta$ or b_ε. The former case contradicts the reducedness and the latter case contradicts the minimality of $\widetilde{L}$ (diagram V). Therefore, if $w(f_{\mathcal{D}})$ $+ w(T_{\mathcal{D}}) < 0$, then $f_{\mathcal{D}}$ is X_6^{ab} or $Y_{8,4}^{ab,\cdot}$. In the first case, let $v_1v_2v_3v_4v_5v_6$ be the boundary cycle of X_6^{ab} and let X_6^{ab} be A-adjacent to T' at $b_1 = b_{\gamma'}$. From the almost alternating property, we have that $b_2 =$ b_α or $b_4 = b_\alpha$. Since the length of curve $b_1b_2b_3b_4b_5b_6$ is 6, we have that $b_2 = b_\alpha$ and $b_4 = b_d$. Then we have that $b_5 = b_\delta$, b_ε or that $b_6 = b_\delta$, b_ε. The first and the fourth cases contradict the reducedness. In the second and the third case, we obtain the diagram of type Γ and type Δ, respectively. For the case of $Y_{8,4}^{ab,\cdot}$, let $v_dv_1v_2v_3v_4v_5v_6v_7$ be the boundary cycle of $Y_{8,4}$ and let $Y_{8,4}$ be adjacent to f_2 at v_dv_1. Then, we do not have that $b_\beta = b_3$ or b_5, since the length of curve $b_db_1b_2b_3b_4b_5b_6b_7$ is 8 and it must pass arc $b_\beta b_\alpha$ and bubble b_γ. If $b_3 = b_\beta$, then we have that $b_5 = b_{\gamma'}$ considering the length of the curve. In the former case, we have that $b_6 = b_\delta$, $b_6 = b_\varepsilon$, $b_7 = b_\delta$, or $b_7 = b_\varepsilon$. The second and the third cases contradict the mimimality of $\widetilde{L}$ from Lemma 4.1. In the first and the fourth case, we obtain T of type Δ and Γ, respectively. (ii) $\sim$ (iv) Note that $T_{\mathcal{E}}$ is T of type Γ, and $T_{\mathcal{F}}$ and $T_{\mathcal{I}}$ are type Δ. Now it is easy to see that those statements hold. (v) We can see this from the primeness and the reducedness of $\widetilde{L}$. ∎

FIGURE 24. Γ and Δ

If $w(f_k) + w(T_k) \geq 0$, then discharge 2 out of the weight of $w(f_k)$ to the block of type T_k, where k is $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{I}$, or $\mathcal{J}$. If every such a sum is non-negative, then we are done. If there is a case that the sum is negative (we know that it is -2 from the above claim), then we go on to the next step.

Step 2: Take weight 2 of the block of type T which the block of type f_k is A-adjacent to (thus its weight goes dowm from 0 to -2) and give it to the block of type T_k, where k is $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, or $\mathcal{I}$. We can see from the proof of the previous claim that the boundary curves of the bottom faces of all the blocks of type T which the blocks of type f_k are A-adjacent to pass the same four bubbles and that such a block of type T is also type Γ or Δ. Now take a look at a face which is B_b-adjacent to a block of type Γ or Δ. Following the proof of Claim 4.11 (ii) and (iii), the only case that the sum of the weights of the block and blocks of type Γ or Δ which it is B_b-adjacent to is negative is that the block is X_6^{ab} and then we obtain a block of type Γ or Δ again, which the block is A-adjacent to.

Step 3: Now, we are at the beginning of Step 2. Since G is a finite graph, we can finally reach the situation so that we can discharge weight of a block to a block of type Γ or Δ with negative weight by continuing this process.

Case 2.

In this case, we look at faces which are C_*-adjacent to blocks of type U, where we use the notation C_* to mean C_l or C_r. From Claim 4.2, we can see that every face can be C_*-adjacent to at most one block of type U at most once. Then we have 7 types of blocks which are C_*-adjacent to blocks of type U; X_i^*, $Y_{i,j}^{p,q}$, and $Z_{i,j}^{p,q}$ with $\{p,q\} = \{\cdot,*\}$, $\{*,\cdot\}$, or $\{*,*\}$, where $*$ stands for l or r. Here, we have the following claim.

Claim 4.12. (i) *No face of degree* 4 *can be* C_**-adjacent to a block of type* U*.*
(ii) *No face of degree* 4 *can be adjacent to two faces of degree* 2 *with any face of degree* 6 *which is* C_**-adjacent to a block of type* U*.*
(iii) *No face of degree* 6 *can be adjacent to two faces of degree* 2 *with any other face of degree* 6 *which is* C_**-adjacent to a block of type* U*.*

Proof. (i) Let $v_\zeta v_\eta v_1 v_2$ be the boundary cycle of the face of degree 4. If $v_1 = v_d$, then it contradicts the minimality of $\widetilde{L}$ (diagram VI). If $v_2 = v_d$, then there exists a face which has two black vertices on its boundary, which contradicts Lemma 3.3. (ii), (iii) Let $v_\zeta v_\eta v_1 v_2 v_3 v_4$ be the boundary cycle of the face of degree 6. Similarly to the previous case, we can show the cases of that $v_1 = v_d$ and $v_4 = v_d$. If $v_3 = v_d$, then it also contradicts the minimality of $\widetilde{L}$ (diagram VII). Therefore, we have that $v_2 = v_d$. Now let $v_1 v_2 v_3 v_5$ and $v_1 v_2 v_3 v_6 v_7 v_8$ be the boundary curve of the face of degree 4 and 6 in the statement, respectively. Then, note that each bounday curve is surrounded by that of the face of degree 6 which is C_*-adjacent to a block of type U. If $b_\zeta = b_5$, b_6, or b_8, then it contradicts the minimality of $\widetilde{L}$ (diagram VI or VII). If $b_7 = b_\zeta$, then it contradicts the primeness. $\square$

If X_i is C_l- and C_r-adjacent to a block of type U, then we call it X_i^l and X_i^r, respectively. Since we do not have X_4^* from Claim 4.12, $w(X_i^*) = i - 4 \geq 6 - 4 = 2$. Similarly we have that $Y_{i,j}^{*,\cdot} = Y_{\geq 6,\geq 4}$, $Y_{i,j}^{\cdot,*} = Y_{\geq 4,\geq 6}$, $Y_{i,j}^{*,*} = Y_{\geq 6,\geq 6}$, $Z_{i,j}^{*,\cdot} = Z_{\geq 8,\geq 4}$, $Z_{i,j}^{\cdot,*} = Z_{\geq 8,\geq 6}$, and $Z_{i,j}^{*,*} = Z_{\geq 8,\geq 6}$. Therefore we have that $w(Y_{i,j}^{*,\cdot}) = i + j - 10 \geq 0$, $w(Y_{i,j}^{\cdot,*}) \geq 0$, $w(Y_{i,j}^{*,*}) \geq 2$, $w(Z_{i,j}^{*,\cdot}) = i + j - 12 \geq 0$, $w(Z_{i,j}^{\cdot,*}) \geq 2$, and $w(Z_{i,j}^{*,*}) \geq 2$. Then, for each block which is C_*-adjacent to blocks of type U, discharge 2 out of its weight to each of the blocks of type U if the sum of the weights of the block and all the blocks of type U is non-negative. The type of block such that the sum is negative is $Y_{6,4}^{*,\cdot}$, $Y_{4,6}^{\cdot,*}$, $Y_{6,6}^{*,*}$, $Z_{8,4}^{*,\cdot}$, or $Z_{8,6}^{*,*}$. For each of these blocks, we say that it is type II if it is C_l- and C_r-adjacent to two blocks of type U. Otherwise, we say that it is type I. Then, we have the following claim, where we say that a block of type U is D_l- or D_r-adjacent to a face if the boundary curves of the block of type U and the block containing the face form diagram Θ shown in Figure 25.

Figure 25. Θ

Claim 4.13. *For every block of type I, the block of type U which it is C_l- (resp. C_r-) adjacent to is D_l- (resp. D_r-) adjacent to a face of the block.*

Proof. The block of type I is $Y_{6,4}^{*,\cdot}$, $Y_{4,6}^{\cdot,*}$, $Y_{6,6}^{l,l}$, $Y_{6,6}^{r,r}$, $Z_{8,4}^{*,\cdot}$, $Z_{8,6}^{l,l}$, or $Z_{8,6}^{r,r}$. We show only the cases of $Y_{6,4}^{l,\cdot}$, $Y_{6,6}^{l,l}$, $Z_{8,6}^{l,l}$. In the case of $Y_{6,4}^{l,\cdot}$, let $v_\zeta v_\eta v_1 v_2 v_3 v_4$ be the boundary curve of Y_6^l. We can show the cases that $v_1 = v_d$ and $v_4 = v_d$ similarly to Claim 4.12 (i). Assume that $v_2 = v_d$. Considering the face of degree 4 of $Y_{6,4}$, we can see that it contradicts the minimality (VI or VII) or primeness of $\widetilde{L}$ in both cases that two faces of $Y_{6,4}$ are adjacent to a face of degree 2 at $v_1 v_2$ and $v_2 v_3$. Next assume that $v_3 = v_d$. Then the two faces of $Y_{6,4}$ are adjacent to a face of degree 2 at $v_2 v_3$ or $v_3 v_4$. The latter case contradicts the minimality of $\widetilde{L}$ (diagram VII). In the former case, let $v_2 v_3 v_5 v_6$ be the boundary cycle of the face of degree 4. Then, we have that $b_5 = b_\eta$, $b_5 = b_\theta$, $b_6 = b_\eta$, or $b_6 = b_\theta$. The first two cases contradict the primeness and the last case contradicts the minimality of $\widetilde{L}$ (diagram VI). In the third case, we obtain diagram Θ.

In the case of $Y_{6,6}^{l,l}$, considering the face of degree 6 whose boundary curve passes the leftside of the dealternator on the diagram and following the previous case, we have that $v_3 = v_d$ and the two faces of degree 6 are adjacent to a face of degree 2 at $v_2 v_3$. Now let $v_2 v_3 v_5 v_6 v_7 v_8$ be the boundary curve of the other face of degree 6. Since it is also C_l-adjacent to a block of type U, we have that b_6 or $b_8 = b_\zeta$ from the almost alternating property. The former case contradicts the primeness. In the latter case, we obtain diagram Θ.

In the case of $Z_{8,6}^{l,l}$, considering the face of degree 6 and following the proof of Claim 4.12 (ii) and (iii), we have that $v_2 = v_d$. Now let $v_1 v_2 v_3 v_5 v_6 v_7 v_8 v_9$ be the boundary cycle of the face of degree 8. Since the face is also C_l-adjacent to a block of type U, we have that b_6 or $b_8 = b_\eta$ considering the

almost alternating property. The former case contradicts the mimimality of $\widetilde{L}$ (diagram VII). In the latter case, we obtain diagram Θ. $\square$

For each of $Y_{6,6}^{*,*}$ and $Z_{6,8}^{*,*}$ of type I, discharge its weight 2 to the block of type U which is not D_l- or D_r-adjacent to any face of the block. Then, we may conclude that if we still have a block of type U with negative weight, then it is D_l- (resp. D_r-) adjacent to a block with weight 0, or it is C_l- (resp. C_r-) adjacent to a block with weight 2 which is C_r- (resp. C_l-) adjacent to another block of type U with negative weight. Then, we can constract finite paths by regarding blocks of type U and blocks of type I and II as edges and vertices, respectively. However then, clearly from their diagrams, if there exists a block of type U which is D_l-adjacent to a face of a block of type I, then there does not exist any block of type U which is D_r-adjacent to a face of a block of type I. Therefore, there does not exist such a path, since its ends should come from blocks of type I.

Case 3.

Now we have a block of type T and a block of type U. If $b_\varepsilon = b_\eta$ or $b_\alpha = b_\theta$, then it contradicts the minimality of $\widetilde{L}$ (diagram III or VI). If $b_\alpha = b_\eta$, then it contradicts the primeness. And if $b_\varepsilon = b_\theta$ and $b_\delta \neq b_\eta$ (or $b_\varepsilon \neq b_\theta$ and $b_\delta = b_\eta$), then it also contradicts the primeness. If $b_\varepsilon = b_\theta$ and $b_\delta = b_\eta$, then we treat the block of type T as a block of type U. In the case that $v_2 = v_d$ of Claim 4.12 (i), we obtain a non-reduced diagram. If we have that $b_\varepsilon = b_\theta$ and $b_\delta = b_\eta$ for every block of type T, then we are done by following Case 2. Therefore, we may assume that there exists at least one block of type T such that $b_\delta \neq b_\eta$ and $b_\varepsilon \neq b_\theta$. Since the diagram shown in Figure 26 and its mirror image cannot coexist, we may assume that T and U replaced on the diagram are shown in Figure 26. First of all, take a look at the blocks which are A-adjacent to blocks of type T and discharge 2 out of its weight to each of the blocks of type T if the sum of the weights of the block and all the blocks of type T is non-negative. After discharging, we have the two cases; there are no blocks of type T with negative weight (**Case 3-1**) and there exists a block of type T with negative weight (**Case 3-2**).

FIGURE 26. T with U

Case 3-1.

Take a look at the blocks which are C_*-adjacent to blocks of type U. Now we have the following claim.

Claim 4.14. *Graph G does not have any blocks of type X_4^*, $Y_4^{\cdot}$, Z_6^*, $Z_{6,i}^{a,*}$.*

Proof. From Claim 4.12, it is sufficient to show only the last three cases. If there exists a block of type $Y_4^{\cdot}$, then it contradicts the primeness, the reducedness, or the minimality of $\widetilde{L}$ (diagram VI or VII). Let $v_d v_1 v_2 v_3 v_4 v_5$ be the boundary cycle of Z_6^*. Cosidering the length of the curve, we can assume that $b_1 = b_\alpha$, $b_1 = b_\beta$, $b_5 = b_\delta$, or $b_5 = b_\varepsilon$. Then, it contradicts the primeness or the minimality of $\widetilde{L}$ (diagram VI or VII). Take a look at a block of type Z_6^a and let $v_\beta v_\gamma v_\delta v_1 v_2 v_3$ be its boundary cycle. From the minimality of $\widetilde{L}$ (diagram VI or VII), we obtain that $v_2 = v_d$ and thus the curve surrounds the boundary curve of the face f which is adjacent to two faces of degree 2 with Z_6^a. Therefore, face f cannot be C_*-adjacent to a block of type U. $\square$

Note that no face can be A-adjacent to T and C_*-adjacent to U. Thus we have 11 types of blocks which are C_*-adjacent to blocks of type U; X_i^*, $Y_{i,j}^{p,q}$, and $Z_{i,j}^{p,q}$ with $\{p, q\} = \{\cdot, a, *\}$ and p or $q = *$.

Then, for each block which is C_*-adjacent to blocks of type U, discharge 2 out of its weight to each of the blocks of type U if the sum of the weights of the block and all the blocks of type U is non-negative. From Claim 4.14, The type of block such that the sum is negative is $Y^{*,a}_{6,6}$, $Y^{a,*}_{6,6}$, or $Y^{*,*}_{6,6}$.

Now let us take a look at a block of type Y^*_6. First, let $v_\zeta v_\eta v_1 v_2 v_3 v_4$ be the boundary cycle of Y^l_6. Then we have that $v_1 \neq v_d$ and $v_4 \neq v_d$ from the proof of Claim 4.12 (i). Assume that $v_2 = v_d$. If $b_1 = b_\delta$, then it contradicts the minimality of $\widetilde{L}$ (diagram VI). Thus we have that $b_1 = b_\varepsilon$ and then the block is adjacent to a face of degree 2 at $v_2 v_3$, otherwise it contradicts the primeness. Therefore we have that $b_3 \neq b_\alpha$, $b_3 \neq b_\beta$, and $b_4 \neq b_\alpha$ from the primeness and the minimality of $\widetilde{L}$ (diagram VII). Thus we have that $b_4 = b_\beta$ and then we obtain diagram D_1 in Figure 27. Similary we obtain D_2 in the case that $v_3 = v_d$. Next, let $v_\theta v_\eta v_1 v_2 v_3 v_4$ be the boundary cycle of Y^r_6. Following the previous case, we see that the case that $v_2 = v_d$ contradicts the primeness and we obtain diagram D_3 in the case that $v_3 = v_d$.

FIGURE 27. Y^l_6 with Y^r_6

Then we can see that we do not have a block of type $Y^{l,l}_{6,6}$ from diagrams D_1 and D_2 paying attention to the boundary curves of faces of degree 2. And we also do not have a block of type $Y^{r,r}_{6,6}$, since the boundary curve of Y^r_6 must pass the rightside of the dealternator on the diagram. In addition, note that the diagrams of Y^l_6 of $Y^{a,*}_{6,6}$ and $Y^{*,a}_{6,6}$ must be D_1, since b_γ is surrounded by the boundary curve of Y^l_6 in diagram D_2. Now we can constract finite paths by regarding the blocks of type U as edges and $Y^{r,l}_{6,6}$, $Y^{l,r}_{6,6}$, $Y^{a,*}_{6,6}$, and $Y^{*,a}_{6,6}$ as vertices. Then, their ends must come from $Y^{*,a}_{6,6}$ or $Y^{a,*}_{6,6}$ and thus we have diagrams D_1 and D_3. However, we can see that it does not happen from Figure 27. Therefore we can conclude that we do not have such paths.

Case 3-2.

In this case, we see that the boundary curves of the bottom faces of blocks of type T with negative weight pass the same four bubbles from the following claim and Claim 4.4. In the rest of this case, we use only this fact and we do not care about the type of a block which is A-adjacent to T.

Claim 4.15. *After discharging, the weight of any block of type T is non-negative, or G contains $\mathcal{A}$, $\mathcal{G}^*$, or $\mathcal{H}$.*

Proof. The diagram obtained from any block of N' and N contains S and S', respectively. It contradicts the minimality of $\widetilde{L}$ (diagram VI or VII), since we have U as well. The diagram obtained from $\mathcal{B}$ (resp. $\mathcal{C}$) contains an arc connecting b_α and b_ε (resp. b_β and b_ε). However, since b_α (resp. b_β) is in one of the two regions of $S^2 - b_d b_\zeta b_\eta b_\theta$ and b_ε is in the other, no arc can connect them. □

Take a look at blocks which are B_b-adjacent to T or C_*-adjacent to U. Then we have the following.

Claim 4.16. *Graph G does not have any block of type X^b_4, X^b_6, X^{b*}_8, Z^{ab}_8, $Y^{ab,*}_{8,6}$, $Y^{ab,a}_{8,6}$, $Y^{ab,ab}_{8,8}$, or $Z^{b,a}_{i,6}$.*

Proof. We omit the proof for the first three and the last cases. Take the boundary cycle $v_\delta v_\gamma v_\beta v_1 v_2 v_3 v_4 v_5$ of Y^{ab}_8. From the almost alternating property and the length of the curve, Y^{ab}_8 is B_b-adjacent to T at $v_\beta v_1$ or $v_2 v_3$. Since the first case contradicts the minimality of $\widetilde{L}$ (diagram VII), we may consider the second case. Then we have that $v_5 = v_d$ from the minimality of $\widetilde{L}$ (diagram VII). Then, we can see that the boundary curve passes the leftside of the dealternator. Therefore, we do not have $Y^{ab,ab}_{8,8}$. Considering the diagram, it is also easy to see that we do not have Z^{ab}_8, $Y^{ab,*}_{8,6}$ or $Y^{ab,a}_{8,6}$. □

We know that no face can be A-adjacent to T and C_*-adjacent to U, that any face can be C_*-adjacent to at most one U at most once and, from the above fact, that any face can be B_b-adjacent to at most one T at most once. Thus we have 68 types of blocks which are B_b-adjacent to T or C_*-adjacent to U; X_i^p, $Y_{i,j}^{q,r}$, and $Z_{i,j}^{q,r}$ with $\{q, r\} = \{\cdot, a, *, b, ab, *b\}$ and p and one of q and r are of $\{*, b, *b, ab\}$. Then, for each block which is C_*-adjacent to a block of type U or B_b-adjacent to a block of type T, discharge 2 out of its weight to each of the blocks of type T and U if the sum of the weights of the block and all the blocks of type T and U is non-negative. From Claim 4.14 and Claim 4.16, we have that the type of a block such that the sum is negative is $Y_{6,6}^{*,a}$, $Y_{6,6}^{a,*}$, or $Y_{6,6}^{*,*}$. However, then we can conclude that the sum of the weights of all faces is non-negative following the previous case.

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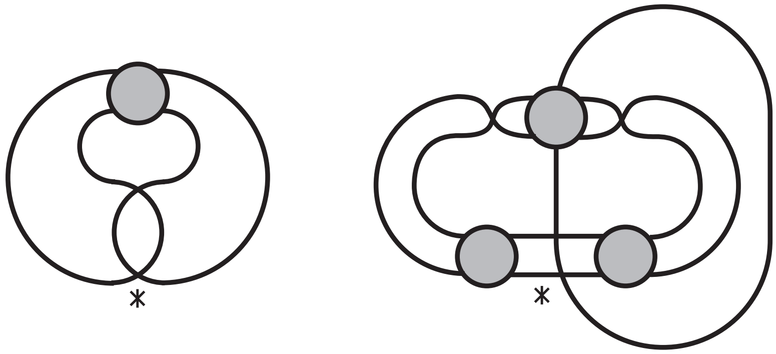
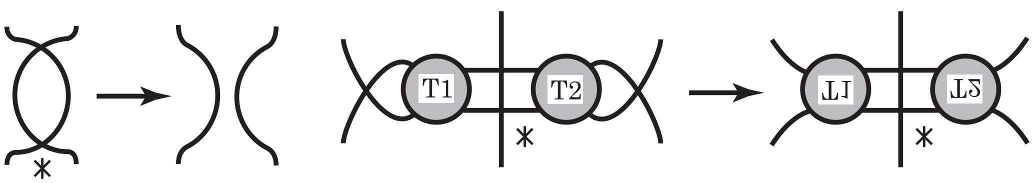


Diagram I Diagram II

Figure 1.



Reducing move I Reducing move II

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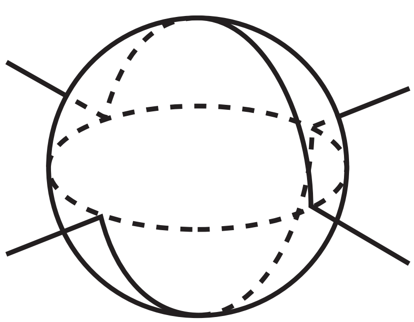

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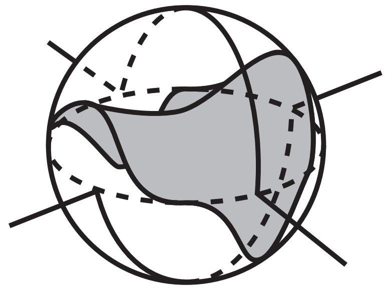

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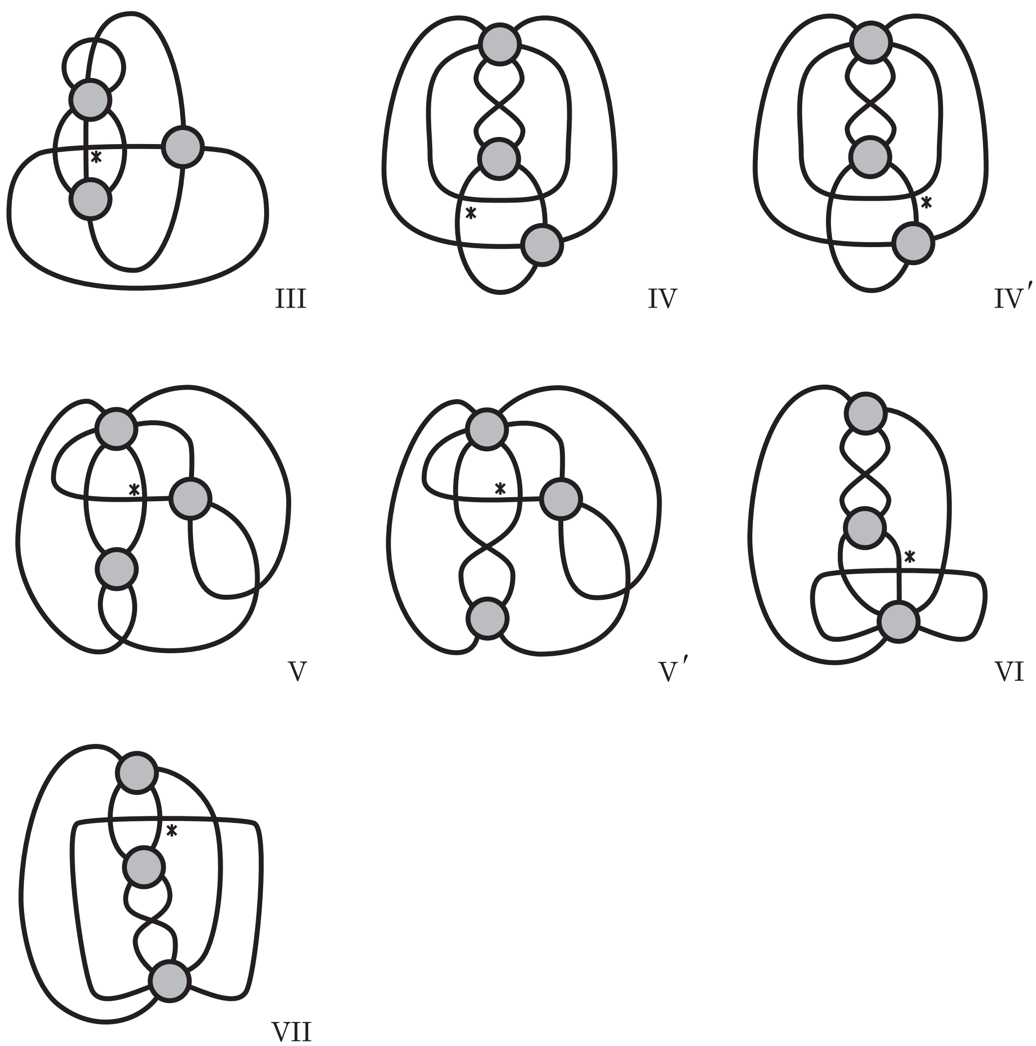

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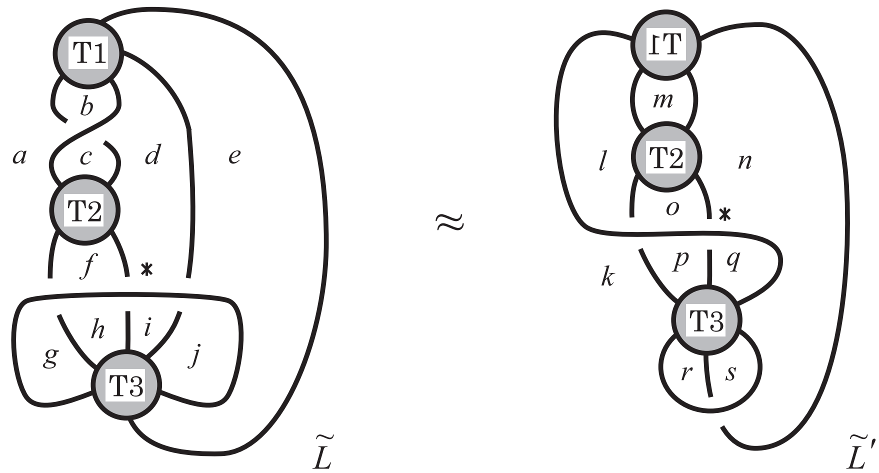

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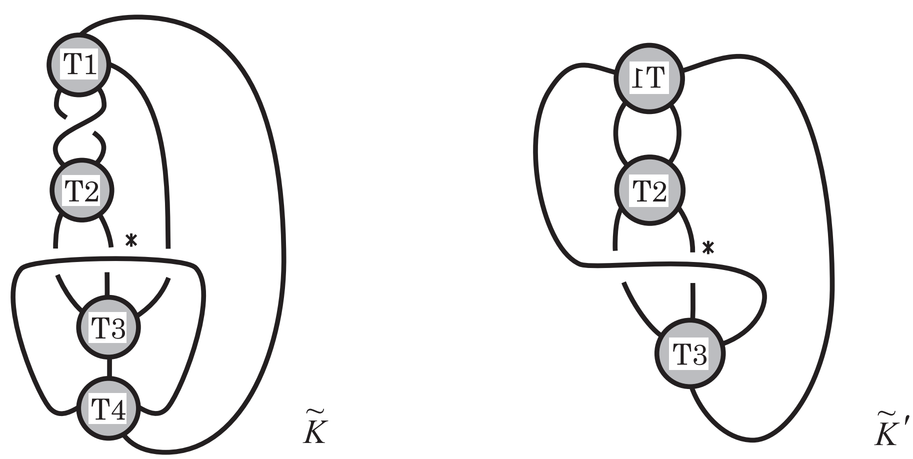

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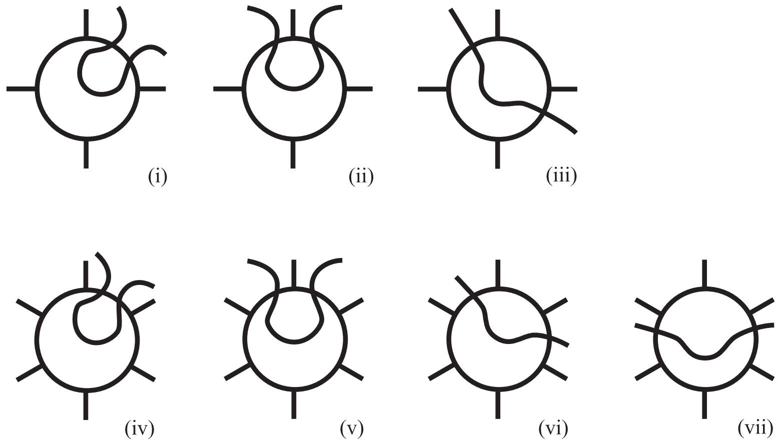

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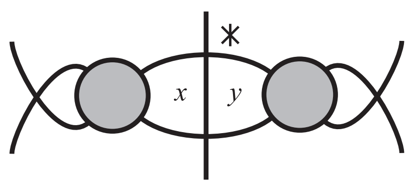

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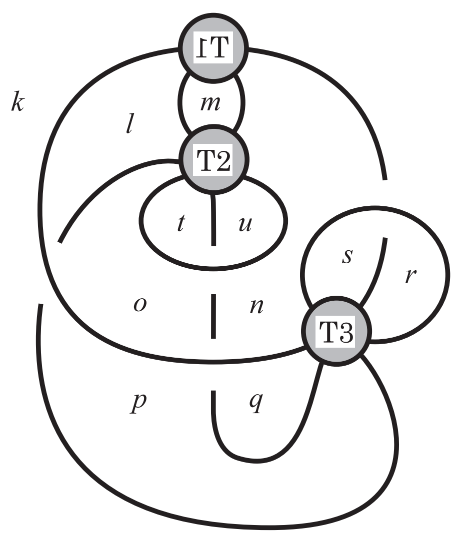

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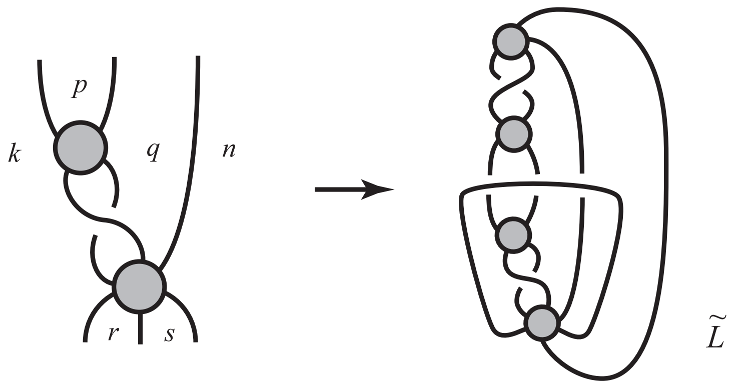

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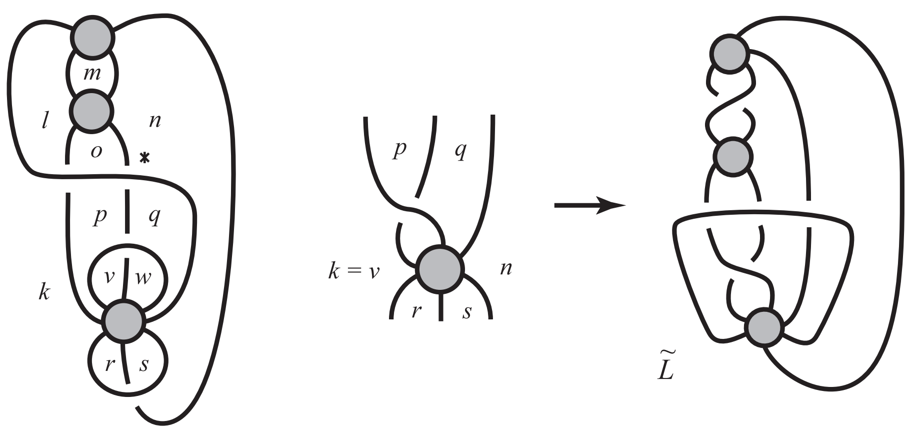

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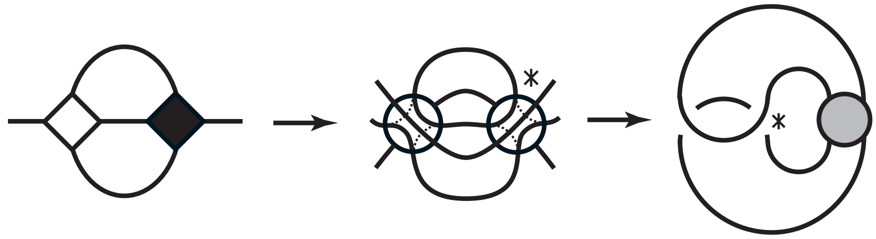

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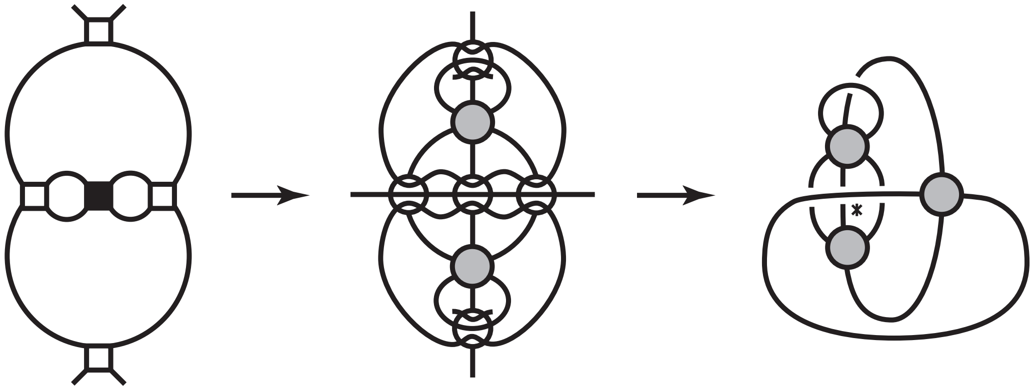

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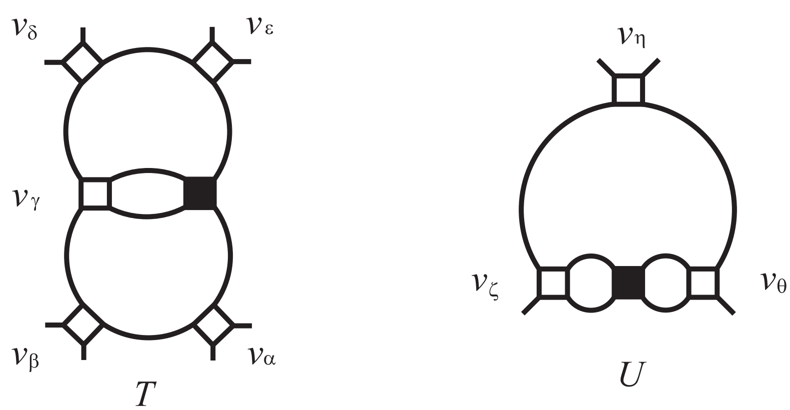

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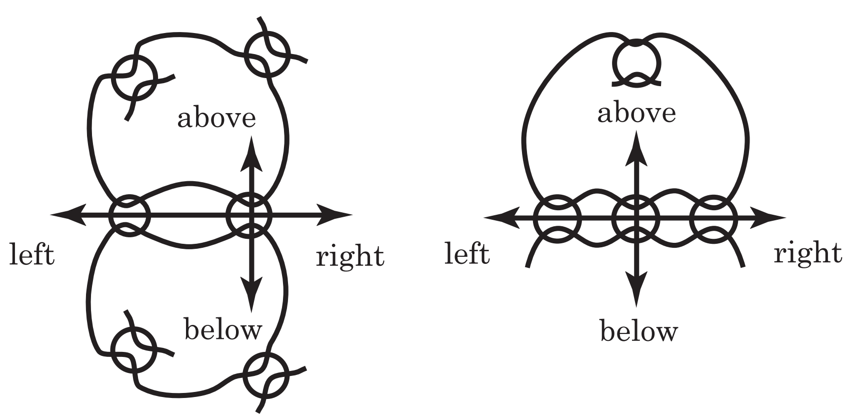

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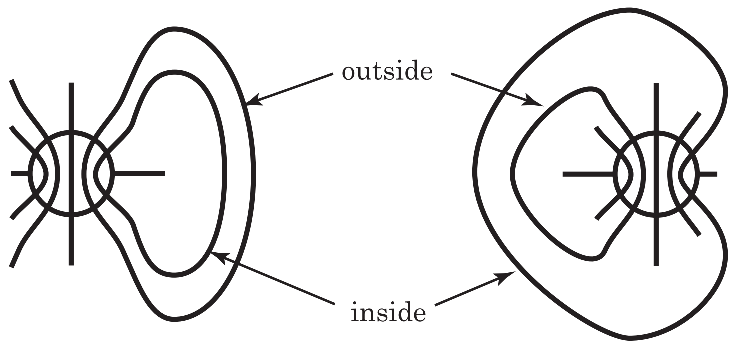

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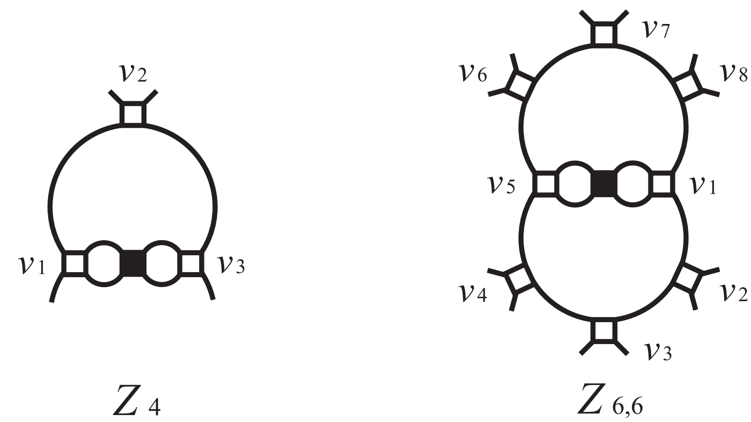

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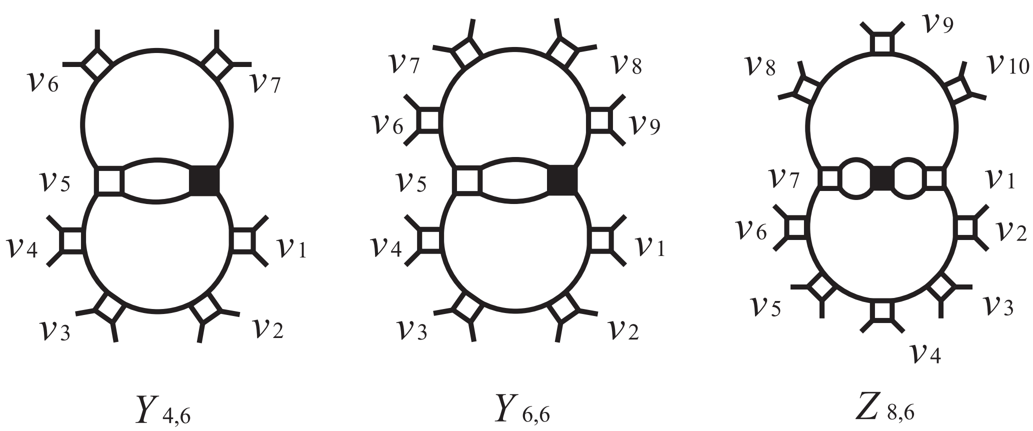

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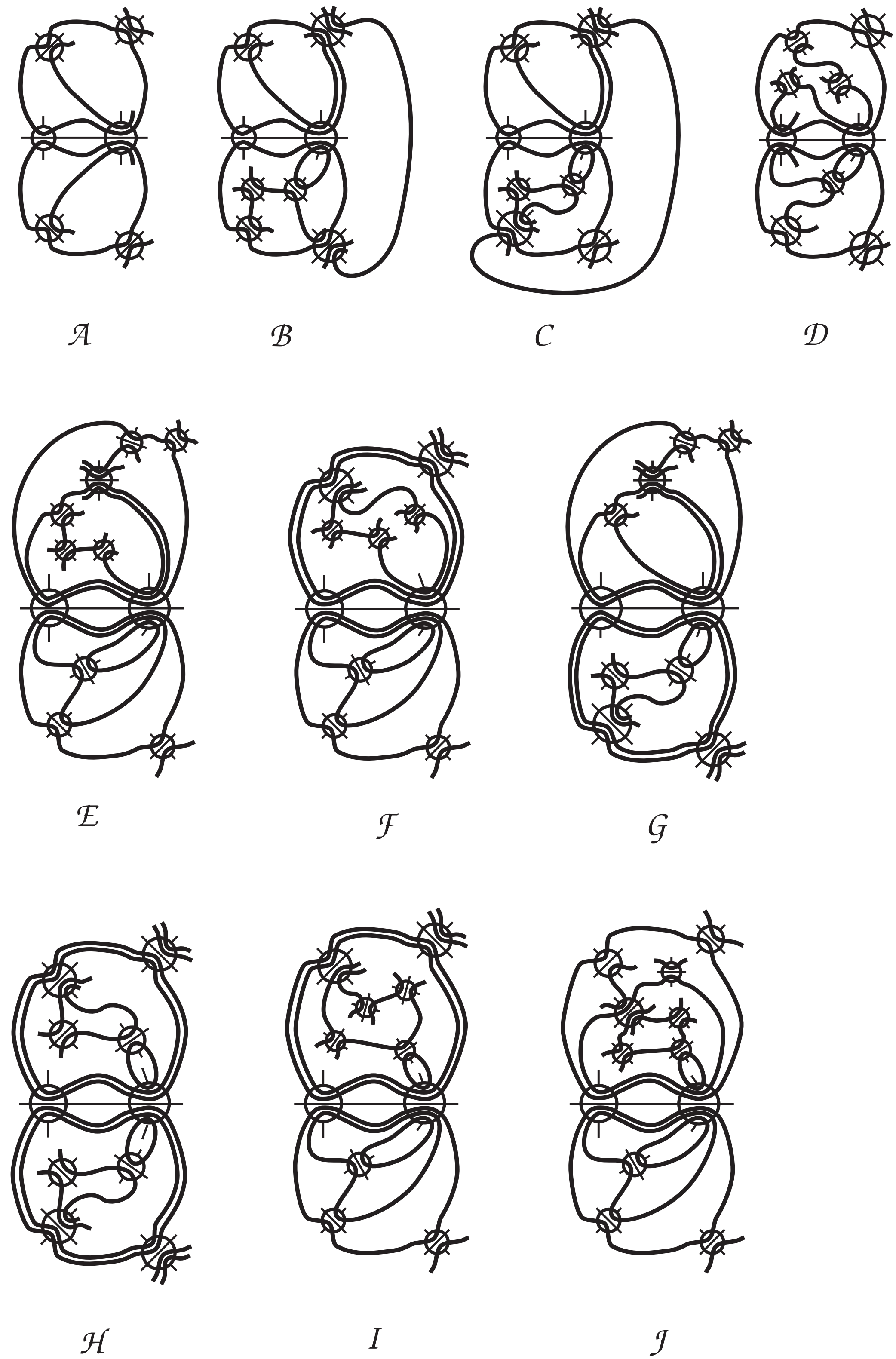

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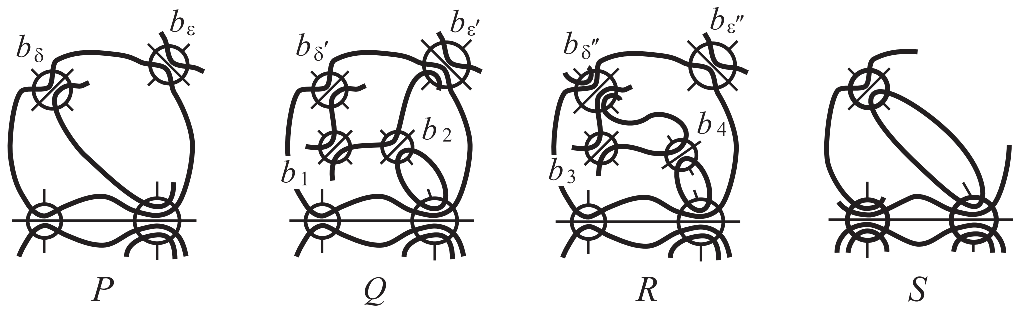

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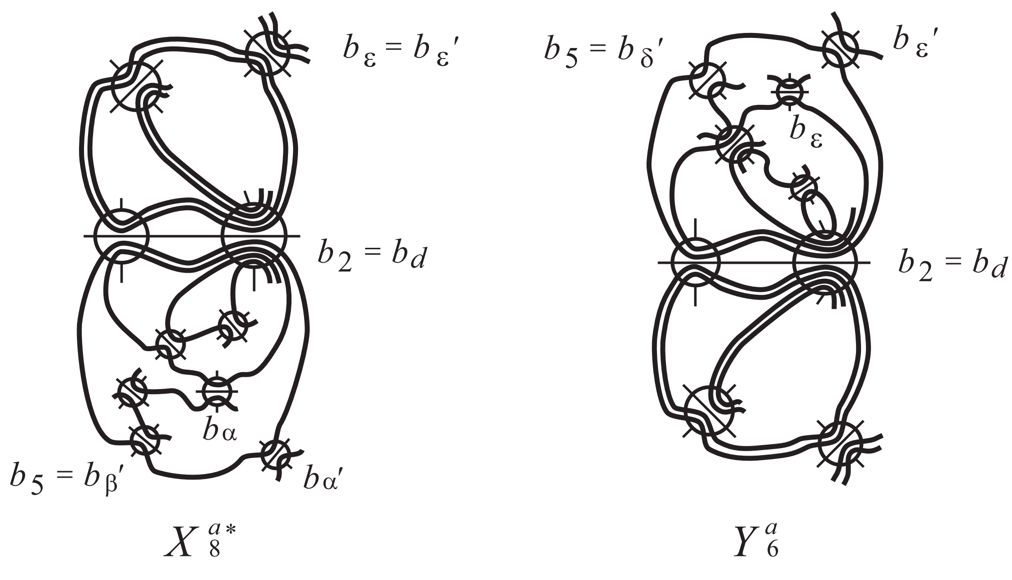

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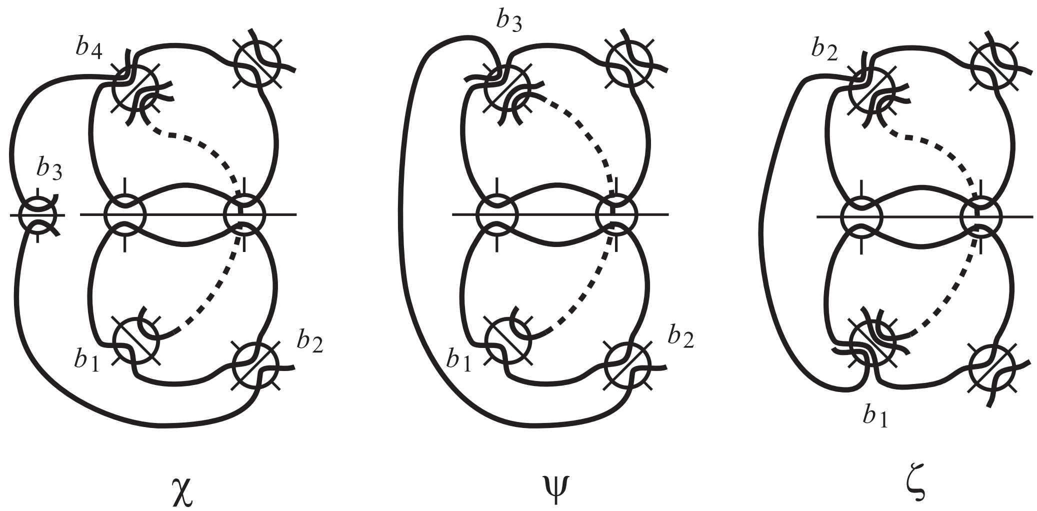

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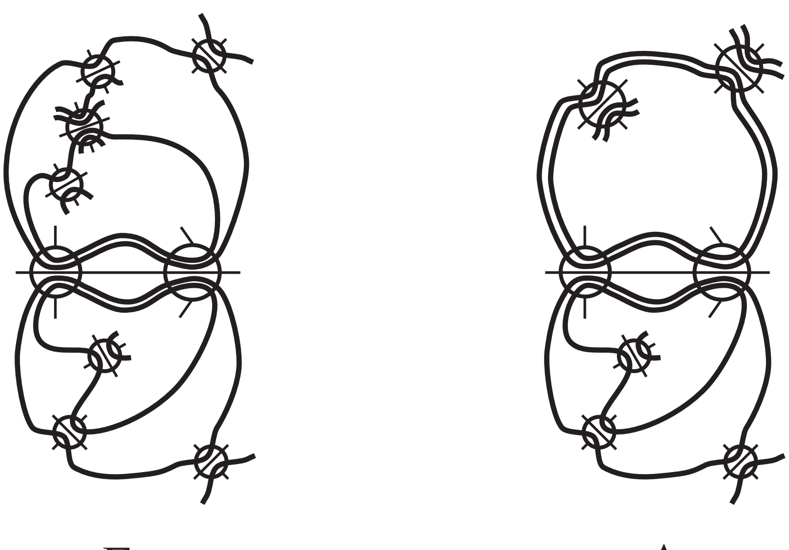

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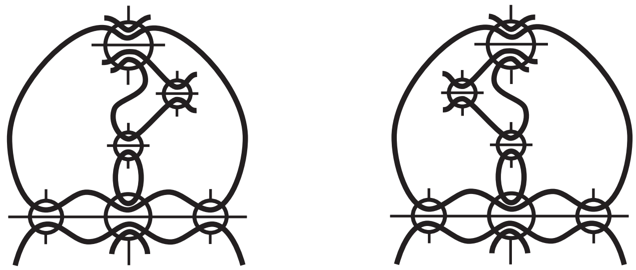
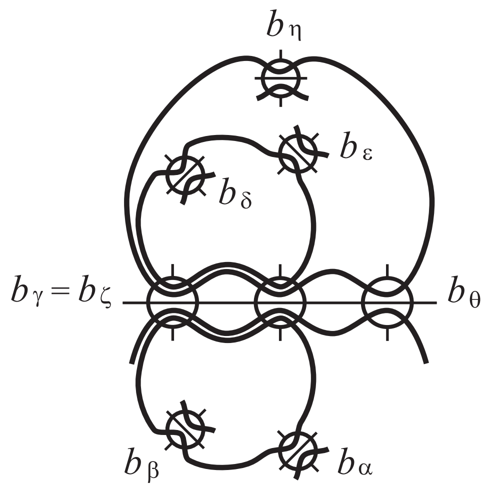


D_r - adjacent D_l - adjacent

Θ

Figure 25.



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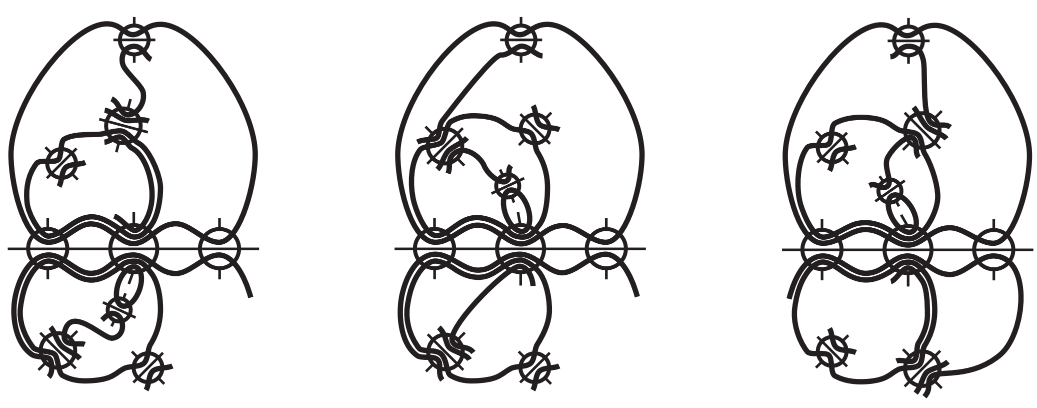

D_1 D_2 D_3

Figure 27.